Common to All Branches
1 & 2 Semester
Chemistry Cycle
2010 Scheme

Engineering Mathematics – 2

[10MAT21]

Compiled by studyeasy.in

Download Notes, Question Banks and other Study Material
www.studyeasy.in
**Branch Name**: Common to all branches  
**SEM**: 1/2  
**University**: VTU  
**Syllabus**: 2010

**Table of Contents:**

**Engineering Mathematics -2 (10MAT21):**

<table>
<thead>
<tr>
<th>Sl. No.</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Differential Equations 1</td>
</tr>
<tr>
<td>2</td>
<td>Differential Equations 2</td>
</tr>
<tr>
<td>3</td>
<td>Differential Equations 3</td>
</tr>
<tr>
<td>4</td>
<td>Partial Differential Equation</td>
</tr>
<tr>
<td>5</td>
<td>Integral Calculus</td>
</tr>
<tr>
<td>6</td>
<td>Vector Integration</td>
</tr>
<tr>
<td>7</td>
<td>Laplace Transforms - 1</td>
</tr>
<tr>
<td>8</td>
<td>Laplace Transforms - 2</td>
</tr>
</tbody>
</table>

Download notes for other subjects from the link below:

www.studyeasy.in

T&C - All rights reserved-studyeasy.in - 2014
SYLLABUS

ENGINEERING MATHEMATICS-II

SUBJECT CODE: 10 MAT 21

PART A

Unit-I: Differential Equations-1

Equations of first order and higher degree (p-y-x equations), Equations solvable for p, y, x. General and singular solutions, Clairaut’s equation. Applications of differential equations of first order.

Unit-II: Differential Equations-2

Linear differential equations: Solution of second and higher order equations with constant coefficients by inverse differential operator method. Simultaneous differential equations of first order.

Unit-III: Differential Equations-3

Method of variation of parameters, Solution of Cauchy’s and Legendre’s linear equations, Series solution of equations of second order, Frobenius method-simple problems.

Unit-IV: Partial Differential Equations(PDE)

Formation of partial differential equations (PDE) by elimination of arbitrary constants & functions. Solution of non-homogeneous PDE by direct integration. Solution of homogeneous PDE involving derivative with respect to one independent variable only. Solution of Lagrange’s linear PDE. Solution of PDE by the method of separation of variables (first and second order equations)
PART-B

Unit-V: Integral Calculus

Multiple Integrals- Evaluation of double integrals and triple integrals. Evaluation of double integrals over a given region, by change of order of integration, by change of variables. Applications to area and volume-illustrative examples.

Beta and Gamma Functions- Properties and problems

Unit-VI: Vector Integration:

Line integrals- definition and problems, surface and volume integrals - definition. Green’s theorem in a plane, Stoke’s and Gauss divergence theorem (statements only).

Unit-VII: Laplace Transforms -I:

Definitions, transforms of elementary functions, properties, periodic function, unit step function and unit impulse function.

Unit-VIII: Laplace Transforms-II:

Index

- Unit 1: Differential Equations – I....................................07-21
- Unit 2: Differential Equations – II....................................21-42
- Unit 3: Differential Equations – III.................................43-60
- Unit 4: Partial Differential Equations..............................61-72
- Unit 5: Integral Calculus...............................................73-92
- Unit 6: Vector Integration.............................................93-102
- Unit 7: Laplace Transforms – I.................................103-138
- Unit 8: Laplace Transforms – II.................................139-158
UNIT I
DIFFERENTIAL EQUATIONS –I

Introduction:
We are familiar with the solution of differential equations (d.e.) of first order and first degree in this unit we discuss the solution of differential equations of first order but not of first degree. In addition to the general solution and particular solution associated with the d.e, we also introduce singular solution. The d.es of first order but not of first degree are also branded as p-y-x equations.

Differential equations of first order and higher degree
If \( y = f(x) \), we use the notation \( \frac{dy}{dx} = p \) throughout this unit.
A differential equation of first order and \( n^{th} \) degree is the form
\[
A_0 p^n + A_1 p^{n-1} + A_2 p^{n-2} + \ldots + A_n = 0
\]
Where \( A_0, A_1, A_2, \ldots A_n \) are functions of \( x \) and \( y \). This being a differential equation of first order, the associated general solution will contain only one arbitrary constant. We proceed to discuss equations solvable for \( P \) or \( y \) or \( x \), wherein the problem is reduced to that of solving one or more differential equations of first order and first degree. We finally discuss the solution of clairaut’s equation.

Equations solvable for \( p \)
Supposing that the LHS of (1) is expressed as a product of \( n \) linear factors, then the equivalent form of (1) is
\[
p - f_1(x, y) - f_2(x, y) \ldots - f_n(x, y) = 0 \quad \ldots(2)
\]
\[
\Rightarrow p - f_1(x, y) = 0, \ p - f_2(x, y) = 0 \ldots \ p - f_n(x, y) = 0
\]
All these are differential equations of first order and first degree. They can be solved by the known methods. If \( F_1(x, y, c) = 0, F_2(x, y, c) = 0, \ldots F_n(x, y, c) = 0 \) respectively represents the solution of these equations then the general solution is given by the product of all these solution.
Note: We need to present the general solution with the same arbitrary constant in each factor.
1. Solve: \( y \left( \frac{dy}{dx} \right)^2 + x - y \frac{dy}{dx} - x = 0 \)

\[
\frac{dy}{dx} = \frac{y-x}{2y} = \frac{(y-x)\pm \sqrt{(y-x)^2+4xy}}{2y}
\]

\[
p = \frac{(y-x)\pm (x+y)}{2y}
\]

\[
\Rightarrow \, p = \frac{y-x+x+y}{2y} \quad \text{or} \quad p = \frac{y-x-x-y}{2y}
\]

\[
ie, \quad p = 1 \quad \text{or} \quad p = \pm x/y
\]

We have

\[
\frac{dy}{dx} = 1 \Rightarrow y = x + c \quad \text{or} \quad (y-x-c) = 0
\]

Also, \( \frac{dy}{dx} = -\frac{x}{y} \) or \( ydy + xdx = 0 \) \( \Rightarrow \int ydy + \int xdx = k \)

\[
ie, \quad \frac{y^2}{2} + \frac{x^2}{2} = k \quad \text{or} \quad y^2 + x^2 = 2k \quad \text{or} \quad (x^2 + y^2 - c) = 0
\]

Thus the general solution is given by \((y-x-c)(y-cx) = 0\)

2. Solve: \( x(y')^2 - (2x+3y) y' + 6y = 0 \)

\[
\frac{dy}{dx} = 2 \Rightarrow \int dy = 2 \int dx + c \quad \text{or} \quad y = 2x + c \quad \text{or} \quad (y - 2x - c) = 0
\]

Also \( \frac{dy}{dx} = \frac{3y}{x} \) or \( \frac{dy}{y} = 3 \frac{dx}{x} \) \( \Rightarrow \int \frac{dy}{y} = 3 \int \frac{dx}{x} + k \)

\[
ie, \quad \log y = 3 \log x + k \quad \text{or} \quad \log y = 3 \log x + \log c, \quad \text{where} \ k = \log x
\]

\[
ie, \quad \log y = \log (cx^3) \Rightarrow y = cx^3 \quad \text{or} \quad y - cx^3 = 0
\]

Thus the general solution is \((y-2x-c)(y-cx^3) = 0\)
3) Solve \( p(p + y) = x(x + y) \)

Solution: The given equation is, \( p^2 + py - x(x + y) = 0 \)

\[
p = \frac{-y \pm \sqrt{y^2 + 4x(x + y)}}{2}
\]

\[
p = \frac{-y \pm \sqrt{4x^2 + 4xy + y^2}}{2} = \frac{-y \pm (2x + y)}{2}
\]

ie., \( p = x \) or \( p = \frac{-2(y + x)}{2} = -(y + x) \)

We have,

\[
\frac{dy}{dx} = x \Rightarrow y = \frac{x^2}{2} + k
\]

Also, \( \frac{dy}{dx} = -y + x \)

ie., \( \frac{dy}{dx} + y = -x \), is a linear d.e (similar to the previous problem)

\[P = 1, \quad Q = -x; e^{\int P \, dx} = e^x\]

Hence \( ye^x = \int -xe^x \, dx + c\)

ie., \( ye^x = -(xe^x - e^x) + c\), integrating by parts.

Thus the general solution is given by \((2y - x^2 - c)\left[e^x(y + x - 1) - c\right] = 0\)

**Equations solvable for y:**

We say that the given differential equation is solvable for y, if it is possible to express y in terms of x and p explicitly. The method of solving is illustrated stepwise.

Y=f(x, p)

We differentiate (1) w.r.t x to obtain

\[
\frac{dy}{dx} = p = F\left(x, y, \frac{dp}{dx}\right)
\]

Here it should be noted that there is no need to have the given equation solvable for y in the explicit form(1). By recognizing that the equation is solvable for y, we can proceed to differentiate the same w.r.t x. We notice that (2) is a differential equation of first order in p and x. We solve the same to obtain the solution in the form \( \phi(x, p, c) = 0 \)

By eliminating p from (1) and (3) we obtain the general solution of the given differential equation in the form \( G(x,y,c) = 0 \)
Remark: Suppose we are unable to eliminate p from (1) and (3), we need to solve for x and y from the same to obtain.

\[ x = F_1(p, c), \quad y = F_2(p, c) \]

Which constitutes the solution of the given equation regarding p as a parameter.

**Equations solvable for x**

We say that the given equation is solvable for x, if it is possible to express x in terms of y and p. The method of solving is identical with that of the earlier one and the same is as follows.

\[ X = f(y, p) \]

Differentiate w.r.t. y to obtain

\[ \frac{dx}{dy} = \frac{1}{p} = F\left(x, y, \frac{dp}{dy}\right) \]

(2) being a differential equation of first order in p and y the solution is of the form.

\[ \phi(y, p, c) = 0 \]

By eliminating p from (1) and (3) we obtain the general solution of the given d.e in the form

\[ G(x, y, c) = 0 \]

Note: The content of the remark given in the previous article continue to hold good here also.

1. **Solve:** \[ y - 2px = \tan^{-1}(xp^2) \]

   \[ \gg By\ data, \quad y = 2px = +\tan^{-1}(xp^2) \]

   The equation is of the form \[ y = f(x, p), \text{ solvable for } y. \]

   Differentiating (1) w.r.t. x,

   \[ p - 2p - 2\frac{dp}{dx}x = \frac{1}{1 + x^2 p^4} \left[ x.2p \frac{dp}{dx} + p^2 \right] \]

   \[ ie., \quad -p - 2x\frac{dp}{dx} = \frac{1}{1 + x^2 p^4} \left[ 2xp \frac{dp}{dx} + p^2 \right] \]
2. Obtain the general solution and the singular solution of the equation \( y + px = p^2 x^4 \)

\[
\text{ie.,} \quad -p - \frac{p^2}{1 + x^2 p^4} = 2x \frac{dp}{dx} \left[ \frac{p}{1 + x^2 p^4} + 1 \right]
\]

\[
\text{ie.,} \quad -p \left[ \frac{1 + x^2 p^4 + p}{1 + x^2 p^4} \right] = 2x \frac{dp}{dx} \left[ \frac{p + 1 + x^2 p^4}{1 + x^2 p^4} \right]
\]

\[
\text{ie.,} \quad \log x + 2 \log p = k
\]

consider \( y = 2px + \tan^{-1}(xp^2) \)

\[
\text{and} \quad xp^2 = c
\]

Using (2) in (1) we have,

\[
y = 2\sqrt{c/x}x + \tan^{-1}(c)
\]

Thus \( y = 2\sqrt{cx} + \tan^{-1}c \), is the general solution.

2. Obtain the general solution and the singular solution of the equation \( y + px = p^2 x^4 \)

\[
\text{The given equation is solvable for y only.}
\]

\[
y + px = p^2 x^4
\]

Differentiating w.r.t x,

\[
\text{ie.,} \quad -2p = x \frac{dp}{dx} + \frac{dx}{x} = -\frac{dp}{2p} \Rightarrow \int \frac{dx}{x} + \frac{1}{2} \int \frac{dp}{p}
\]

\[
\text{ie.,} \quad \log x + \log \sqrt{p} = k \quad \text{or} \quad \log (x\sqrt{p}) = \log c \Rightarrow x\sqrt{p} = c
\]

Consider, \( y + px = p^2 x^4 \)

\[
x\sqrt{p} = c \quad \text{or} \quad x^2 p = c \quad \text{or} \quad p = c / x^2
\]

Using (2) in (1) we have, \( y + (c / x^2)x = (c^2 / x^4)x^4 \)

Thus \( xy + c = c^2 x \) is the general solution.

Now, to obtain the singular solution, we differentiate this relation partially w.r.t c, treating c as a parameter.
That is, 1 = 2cx or c = 1/2x.

The general solution now becomes,

\[ xy + \frac{1}{2x} = \frac{1}{4x^2} x \]

Thus \( 4x^2y + 1 = 0 \) is the singular solution.

3) Solve \( y = p \sin p + \cos p \)

\[ \text{Differentiating w.r.t. } x, \]

\[ p = p \cos p \frac{dp}{dx} + \sin p \frac{dp}{dx} - \sin p \frac{dp}{dx} \]

**ie.,** \( 1 = \cos p \frac{dp}{dx} \) or \( \cos p \frac{dp}{dx} = dx \Rightarrow \cos p \frac{dp}{dx} = \{dx + c\} \)

**ie.,** \( \sin p = x + c \) or \( x = \sin p - c \)

Thus we can say that \( y = p \sin p + \cos p \) and \( x = \sin p - c \) constitutes the general solution of the given d.e.

**Note:** \( \sin p = x + c \Rightarrow p = \sin^{-1}(x + c) \).

We can as well substitute for \( p \) in (1) and present the solution in the form,

\[ y = (x + c)\sin^{-1}(x + c) + \cos \sin^{-1}(x + c) \]

4) Obtain the general solution and singular solution of the equation \( y = 2px + p^2y \).

Solution: The given equation is solvable for \( x \) and it can be written as \( 2x = \frac{y}{p} - py \ldots \ldots (1) \)

Differentiating w.r.t \( y \) we get

\[ \frac{2}{p} = \frac{1}{p} \frac{dp}{dy} - \frac{y}{p^2} \frac{dp}{dy} - \frac{y}{p} \frac{dp}{dy} \]

\[ \Rightarrow \left( \frac{1}{p} + p \right) \left( 1 + \frac{y}{p} \frac{dp}{dy} \right) = 0 \]
Ignoring \(\frac{1}{p} + p\), which does not contain \(\frac{dp}{dy}\), this gives

\[1 + \frac{y}{p} \frac{dp}{dy} = 0 \quad \text{or} \quad \frac{dy}{y} + \frac{dp}{p} = 0\]

Integrating we get

\[yp = c \quad \text{........(2)}\]

Substituting for \(p\) from (2) in (1)

\[y^2 = 2cx + c^2\]

5) Solve \(p^2 + 2py\cot x = y^2\).

Solution: Dividing throughout by \(p^2\), the equation can be written as

\[\frac{y^2}{p^2} = \frac{2y}{p} \cot x = 1\]

Adding \(\cot^2 x\) to b.s

\[\frac{y^2}{p^2} = \frac{2y}{p} \cot x + \cot^2 x = 1 + \cot^2 x\]

or

\[\left(\frac{y}{p} - \cot x\right)^2 = \cos ecx^2\]

\[\Rightarrow \frac{y}{p} - \cot x = \pm \cos ecx\]

\[\Rightarrow \frac{dy}{dx} = \cot x \pm \cos ecx\]

\[\Rightarrow \frac{dy}{y} = \frac{\sin x}{\cos x + 1} \quad \text{and} \quad \frac{dy}{y} = \frac{\sin x}{\cos x - 1}\]

Integrating these two equations we get

\[y(\cos x + 1) = c_1 \quad \text{and} \quad y(\cos x - 1) = c_2\]

General solution is

\[y(\cos x + 1) - c_1 \quad y(\cos x - 1) - c_2 = 0\]

6) Solve: \(p^2 - 4x^5 p - 12x^4 y = 0\), obtain the singular solution also.

Solution: The given equation is solvable for \(y\) only.

\[p^2 - 4x^5 p - 12x^4 y = 0 \quad \text{...........(1)}\]

\[y = \frac{p^2 + 4x^5 p}{12x^4} = f(x, p)\]

Differentiating (1) w.r.t. \(x\),
\[
2p \frac{dp}{dx} + 4x^5 \frac{dp}{dx} + 20x^4 p - 12x^4 p - 48x^3 y = 0
\]
\[
\frac{dp}{dx} (2p + 4x^5) + 8x^3 (xp - \frac{p^2 + 4x^5 p}{2x^4}) = 0
\]
\[
(p + 2x^5) \frac{dp}{dx} = \frac{2p}{x} (p + 2x^5)
\]
\[
\frac{dp}{dx} - \frac{2p}{x} = 0
\]
⇒ Integrating \[ \log p - \log x = k \]
⇒ \[ p = c^2 x^2 \] \[ ; \] equation (1) becomes
\[ c^4 + 4c^2 x^3 = 12y \]

Setting \[ c^2 = k \] the general solution becomes
\[ k^2 + 4kx^3 = 12y \]

Differentiating w.r.t \( k \) partially we get
\[
2k + 4x^3 = 0
\]

Using \( k = -2x^3 \) in general solution we get
\[ x^6 + 3y = 0 \] as the singular solution

7) Solve \[ p^3 - 4xy p + 8y^2 = 0 \] by solving for x.

Solution: The given equation is solvable for \( x \) only.

\[ p^3 - 4xy p + 8y^2 = 0 \]

\[ x = \frac{p^3 + 8y^2}{4yp} = f(y, p) \]

Differentiating (1) w.r.t. \( y \),
\[
3p^2 \frac{dp}{dy} - 4xy \frac{dp}{dy} - 4yp \frac{1}{p} - 4px + 16y = 0
\]

\[
\frac{dp}{dy} (3p^2 - 4xy) = 4px - 12y
\]

\[
\frac{dp}{dy} \left[ 3p^2 \frac{p^3 + 8y^2}{p} \right] = \left[ \frac{p^3 + 8y^2}{y} \right] - 12y
\]

\[
\frac{dp}{dy} \left[ \frac{2p^3 - 8y^2}{p} \right] = \frac{p^3 - 4y^2}{y}
\]

\[
\frac{2}{p} \frac{dp}{dy} (p^3 - 4y^2) = \frac{p^3 - 4y^2}{y}
\]
\[
\frac{2 \, dp}{p \, dy} = \frac{1}{y}
\]
\[
2 \log p = \log y + \log c
\]
\[
U \sin \theta P = \sqrt{cy} \text{ in} \, (1) \, \text{we have,}
\]
\[
cy \sqrt{cy} - 4xy \sqrt{cy} + 8y^2 = 0
\]
Dividing throughout by \( y \sqrt{y} = y^{3/2} \) we have,
\[
c \sqrt{c - 4x} \sqrt{c} + 8 \sqrt{y} = 0
\]
\[
\sqrt{c} \, (c - 4x) = -8 \sqrt{y}
\]
Thus the general solution is \( sc(c - 4x)^2 = 64y \)

**Clairaut’s Equation**

The equation of the form \( y = px + f(p) \) is known as Clairaut’s equation.

This being in the form \( y = F(x, p) \), that is solvable for \( y \), we differentiate (1) w.r.t. \( x \)

\[
\therefore \frac{dy}{dx} = p = p + x \frac{dp}{dx} + f'(p) \frac{dp}{dx}
\]

This implies that \( \frac{dp}{dx} = 0 \) and hence \( p = c \)

Using \( p = c \) in (1) we obtain the general solution of clairaut’s equation in the form
\[
y = cx + f(c)
\]

1. **Solve**: \( y = px + \frac{a}{p} \)

>> The given equation is Clairaut’s equation of the form \( y = px + f(p) \), whose general solution is \( y = cx + f(c) \)

Thus the general solution is \( y = cx + \frac{a}{c} \)

**Singular solution**

Differentiating partially w.r.t \( c \) the above equation we have,
0 = x - \frac{a}{c^2}
\[ c = \sqrt{\frac{a}{x}} \]

Hence \( y = cx + (a/c) \) becomes,
\[ y = \sqrt{a/x} \cdot x + a\sqrt{x/a} \]
Thus \( y^2 = 4ax \) is the singular solution.

2. Modify the following equation into Clairaut’s form. Hence obtain the associated general and singular solutions.

\[ xp^2 - py + kp + a = 0 \]

\[ \Rightarrow xp^2 - py + kp + a = 0, \text{ by data} \]

ie., \( xp^2 + kp + a = py \)

ie., \( y = \frac{p(xp + k) + a}{p} \)

ie., \( y = px + \left( k + \frac{a}{p} \right) \)

Here (1) is in the Clairaut’s form \( y = px + f(p) \) whose general solution is \( y = cx + f(c) \)

Thus the general solution is \( y = cx + \left( k + \frac{a}{c} \right) \)

Now differentiating partially w.r.t c we have,
\[ 0 = x - \frac{a}{c^2} \]
\[ c = \sqrt{a/x} \]

Hence the general solution becomes,
\[ y - k = 2\sqrt{ax} \]

Thus the singular solution is \( (y - k)^2 = 4ax \).

Remark: We can also obtain the solution in the method: solvable for y.
3. Solve the equation \((px - y)(py + x) = 2p\) by reducing into Clairaut’s form, taking the substitutions \(X = x^2\), \(Y = y^2\)
\[
\Rightarrow X = x^2 \quad \Rightarrow \quad \frac{dX}{dx} = 2x
\]
\[
Y = y^2 \quad \Rightarrow \quad \frac{dY}{dy} = 2y
\]
Now, \(p = \frac{dy}{dx} = \frac{dY}{dX} \frac{dX}{dx}\) and let \(p = \frac{dY}{dx}\)
\[ie., \quad p = \frac{1}{2y}.p.2x\]
\[ie., \quad p = \frac{\sqrt{X}}{\sqrt{Y}}.P\]
Consider \((px - y)(py + x) = 2p\)
\[ie., \quad \left[\frac{\sqrt{X}}{\sqrt{Y}}.P\sqrt{X} - \sqrt{Y}\right] \left[\frac{\sqrt{X}}{\sqrt{Y}}.P\sqrt{Y} - \sqrt{X}\right] = 2.\frac{\sqrt{X}}{\sqrt{Y}}.P\]
\[ie., \quad (PX - Y) \quad (P + 1) = 2P\]
\[ie., \quad Y = PX - \frac{2P}{P + 1} is \text{ in the Clairaut’s form and hence the associated general solution is}\]
\[Y = cX - \frac{2c}{c + 1}\]
Thus the required general solution of the given equation is \(y^2 = cx^2 - \frac{2c}{c + 1}\)

4) Solve \(px - y \quad py + x = a^2 p\), use the substitution \(X = x^2\), \(Y = y^2\).
Solution: Let \(X = x^2 \Rightarrow \frac{dX}{dx} = 2x\)
\[
Y = x^2 \Rightarrow \frac{dX}{dy} = 2y
\]
Now, \(p = \frac{dy}{dx} = \frac{dY}{dX} \frac{dX}{dx}\) and let \(P = \frac{dY}{dx}\)
\[P = \frac{1}{2y}.p.2x \quad or \quad p = \frac{x}{y}.P\]
\[p = \frac{\sqrt{X}}{\sqrt{Y}}.P\]
Consider \((px - y)(py + x) = 2p\)

\[
\left( \frac{\sqrt{X}}{\sqrt{Y}} P \sqrt{X} - \sqrt{Y} \right) \left( \frac{\sqrt{X}}{\sqrt{Y}} P \sqrt{Y} + \sqrt{X} \right) = 2 \frac{\sqrt{X}}{\sqrt{Y}} \ P
\]

\((PX - Y)(P + 1) = 2P\)

\[Y = PX - \frac{2P}{P + 1}\]

Is in the Clairaut’s form and hence the associated general solution is

\[Y = cX - \frac{2c}{c + 1}\]

Thus the required general solution of the given equation is

\[y^2 = cx^2 - \frac{2c}{c + 1}\]

5) Obtain the general solution and singular solution of the Clairaut’s equation \(xp^3 - yp^2 + 1 = 0\). (Dec 2011)

Solution: The given equation can be written as

\[y = \frac{xp^3 + 1}{p^2} \Rightarrow y = px + \frac{1}{p^2} \text{ is in the Clairaut’s form } y = px + f(p)\]

whose general solution is \(y = cx + f(c)\)

Thus general solution is \(y = cx + \frac{1}{c^2}\)

Differentiating partially w.r.t. \(c\) we get

\[0 = x - \frac{2}{c^3} \Rightarrow c = \left(\frac{2}{x}\right)^{1/3}\]

Thus general solution becomes

\[y = \left(\frac{2}{x}\right)^{1/3} x + \left(\frac{x}{2}\right)^{2/3} \Rightarrow 2^{2/3} y = 3x^{2/3}\]

or \(4y^3 = 27x^2\)

APPLICATIONS OF DIFFERENTIAL EQUATIONS OF FIRST ORDER

We are familiar with an application of differential equation of first order in the form of Orthogonal Trajectories.

We present a few illustrative problems on Application to Electric Circuits.

A governing first order differential equation by Kirchhoff’s law is given by

\[L \frac{di}{dt} + Ri = E\]

Where \(L\) is the inductance, \(R\) is the resistance and \(E\) is the electromotive force.
1. A constant electromotive force $E$ volts is applied to a circuit containing a constant resistance $R$ ohms in series and a constant inductance $L$ henries. If the initial current is Zero, find the current in the circuit at any time $t$.

We have the governing differential equation,

$$L \frac{di}{dt} + Ri = E$$

This is a linear d.e of the form $\frac{dy}{dx} + py = Q$ whose solution is

$$ye^{\int p \, dx} = \int Qe^{\int p \, dx} \, dx + c$$

Hence the solution of (1) is given by,

$$ie^{\frac{Rt}{L}} = \frac{E}{L} e^{\frac{Rt}{L}} \, dt + c$$

$$ie, \quad ie^{\frac{Rt}{L}} = \frac{E}{L} \frac{e^{\frac{Rt}{L}}}{Rt} + c$$

$$ie, \quad ie^{\frac{Rt}{L}} = \frac{E}{R} e^{\frac{Rt}{L}} + c$$

Using the initial condition, $i=0$ when $t=0$, (2) becomes,

$$0 = \frac{E}{R} + c$$

Using this value of $c$ in (2), we obtain the current in the circuit at any time $t$,

$$i = \frac{E}{R} \left[ 1 - e^{-\frac{Rt}{L}} \right]$$

2. The differential equation for the current in in an electric circuit containing an inductance $L$ and a resistance $R$ in series and acted on by an electromotive force $E \sin wt$ satisfies the equation $L \frac{di}{dt} + Ri = E \sin wt$. Find the value of current at any time $t$, if initially there is no current in the circuit.

The given equation is put in the form,

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L} \sin wt$$

The solution of this linear equation is
\[ i e^{Rt} = \int \frac{E}{L} \sin wt \ e^{Rt/L} \ dt + c \]

\[ i.e., \quad i e^{Rt} = \frac{E}{L} \int e^{Rt/L} \sin wt \ dt + c \]

Note: We are proceed as in the previous problem and obtain the solution. We can also obtain the solution using the following alternative formula for the integral of \( e^{at} \sin bt \).

We have
\[ \int e^{at} \sin bt \ dt = \frac{e^{at}}{\sqrt{a^2 + b^2}} \sin (bt - \tan^{-1} \frac{b}{a}) \]

\[ \therefore \quad i e^{Rt} = \frac{E}{L} \int \frac{e^{Rt}}{\sqrt{R^2 + w^2 L^2}} \sin \left[ wt - \tan^{-1} \frac{wL}{R} \right] + c \]

\[ i.e., \quad i e^{Rt} = \frac{Ee^{Rt} L}{\sqrt{R^2 + w^2 L^2}} \sin \left[ wt - \tan^{-1} \frac{wL}{R} \right] + c \]

Putting \( \varphi = \tan^{-1} \frac{wL}{R} \) we have,

\[ i = \frac{E}{\sqrt{R^2 + w^2 L^2}} \sin wt - \varphi + ce^{-Rt} \]

Using the initial condition, \( I = 0 \) when \( t = 0 \), we have

\[ 0 = \frac{E}{\sqrt{R^2 + w^2 L^2}} \sin(-\varphi) + c \]

Using this value of \( c \) in (2) we have,

\[ i = \frac{E}{\sqrt{R^2 + w^2 L^2}} \left[ \sin wt - \varphi + \sin \varphi e^{-Rt} \right], \text{ where } \varphi = \tan^{-1} \frac{wL}{R} - R \]
UNIT II

DIFFERENTIAL EQUATIONS – II

INTRODUCTION:

We have studied methods of solving ordinary differential equations of first order and first degree, in chapter-7 (1st semester). In this chapter, we study differential equations of second and higher orders. Differential equations of second order arise very often in physical problems, especially in connection with mechanical vibrations and electric circuits.

LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER WITH CONSTANT COEFFICIENTS

A differential equation of the form

\[ \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = X \] ... (1)

where \( X \) is a function of \( x \) and \( a_1, a_2, \ldots, a_n \) are constants is called a linear differential equation of \( n^{th} \) order with constant coefficients. Since the highest order of the derivative appearing in (1) is \( n \), it is called a differential equation of \( n^{th} \) order and it is called linear.

Using the familiar notation of differential operators:

\[ D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2}, \quad D^3 = \frac{d^3}{dx^3}, \ldots, \quad D^n = \frac{d^n}{dx^n} \]

Then (1) can be written in the form

\[ \{D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n\} y = X \]

i.e.,

\[ f(D) y = X \] ... (2)

where

\[ f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n. \]

Here \( f(D) \) is a polynomial of degree \( n \) in \( D \).

If \[ x = 0, \text{ the equation} \]

\[ f(D) y = 0 \]

is called a homogeneous equation.

If \( x \neq 0 \) then the Eqn. (2) is called a non-homogeneous equation.
SOLUTION OF A HOMOGENEOUS SECOND ORDER LINEAR DIFFERENTIAL EQUATION

1. Solve \( \frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + 6y = 0 \).

Solution. Given equation is \( (D^2 - 5D + 6) \ y = 0 \)
A.E. is \( m^2 - 5m + 6 = 0 \)
i.e., \( (m-2)(m-3) = 0 \)
i.e., \( m = 2, 3 \)
\( \therefore m_1 = 2, m_2 = 3 \)
\( \therefore \) The roots are real and distinct.

We consider the homogeneous equation
\[
\frac{d^2 y}{dx^2} + p \frac{dy}{dx} + qy = 0
\]
where \( p \) and \( q \) are constants
\( (D^2 + pD + q) \ y = 0 \)

The Auxiliary equations (A.E.) put \( D = m \)
\( m^2 + pm + q = 0 \)

Eqn. (3) is called auxiliary equation (A.E.) or characteristic equation of the D.E. eqn. (quadratic in \( m \), will have two roots in general. There are three cases.

Case (i): Roots are real and distinct
The roots are real and distinct, say \( m_1 \) and \( m_2 \) i.e., \( m_1 \neq m_2 \)
Hence, the general solution of eqn. (1) is
\[
y = C_1 e^{m_1 x} + C_2 e^{m_2 x}
\]
where \( C_1 \) and \( C_2 \) are arbitrary constant.

Case (ii): Roots are equal
The roots are equal i.e., \( m_1 = m_2 = m \).
Hence, the general solution of eqn. (1) is
\[
y = (C_1 + C_2 x) e^{mx}
\]
where \( C_1 \) and \( C_2 \) are arbitrary constant.

Case (iii): Roots are complex
The Roots are complex, say \( \alpha \pm i\beta \)
Hence, the general solution is
\[
y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)
\]
where \( C_1 \) and \( C_2 \) are arbitrary constants.

Note. Complementary Function (C.F.) which itself is the general solution of the D.E.
The general solution of the equation is
\[ y = C_1 e^{2x} + C_2 e^{3x}. \]

2. Solve \( \frac{d^3 y}{dx^3} - \frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 4y = 0. \)

**Solution.** Given equation is \((D^3 - D^2 - 4D + 4)\) \(y\)

A.E. is \(m^3 - m^2 - 4m + 4 = 0\)
\[ m^3 (m - 1) - 4 (m - 1) = 0 \]
\[ (m - 1) (m^2 - 4) = 0 \]
\[ m = 1, m = \pm 2 \]
\[ m_1 = 1, m_2 = 2, m_3 = -2 \]

\[ \therefore \text{The general solution of the given equation is} \]
\[ y = C_1 e^x + C_2 e^{2x} + C_3 \]

3. Solve \( \frac{d^2 y}{dx^2} - \frac{dy}{dx} - 6y = 0. \)

**Solution.** The D.E. can be written as
\((D^2 - D - 6)\) \(y = 0\)

A.E. is \(m^2 - m - 6 = 0\)
\[ (m - 3) (m + 2) = 0 \]
\[ m = 3, -2 \]
\[ \therefore \text{The general solution is} \]
\[ y = C_1 e^{3x} + C_2 e^{-2x}. \]

4. Solve \( \frac{d^2 y}{dx^2} + 8 \frac{dy}{dx} + 16y = 0. \)

**Solution.** The D.E. can be written as
\((D^2 + 8D + 16)\) \(y = 0\)

A.E. is \(m^2 + 8m + 16 = 0\)
\[ (m + 4)^2 = 0 \]
\[ (m + 4) (m + 4) = 0 \]
\[ m = -4, -4 \]
\[ \therefore \text{The general solution is} \]
\[ y = (C_1 + C_2 x) e^{-4x}. \]

5. Solve \( \frac{d^2 y}{dx^2} + w^2 y = 0. \)

**Solution.** Equation can be written as
\((D^2 + w^2)\) \(y = 0\)

A.E. is \(m^2 + w^2 = 0\)
\[ m^2 = -w^2 = w^2i^2 \quad (i^2 = -1) \]
\[ m = \pm wi \]

This is the form \( \alpha \pm i\beta \) where \( \alpha = 0, \beta = w \).

\[ \therefore \text{ The general solution is } y = e^{\theta t} (C_1 \cos wt + C_2 \sin wt) \]

\[ \therefore y = C_1 \cos wt + C_2 \sin wt. \]

6. Solve \( \frac{d^2y}{dx^2} + 4 \frac{dy}{dx} + 13y = 0 \).

**Solution.** The equation can be written as

\[ (D^2 + 4D + 13)y = 0 \]

A.E. is \( m^2 + 4m + 13 = 0 \)

\[ m = \frac{-4 \pm \sqrt{16 - 52}}{2} \]

\[ = -2 \pm 3i \quad \text{(of the form } \alpha \pm i\beta) \]

\[ \therefore \text{ The general solution is } y = e^{-2x} (C_1 \cos 3x + C_2 \sin 3x). \]

**INVERSE DIFFERENTIAL OPERATOR AND PARTICULAR INTEGRAL**

Consider a differential equation

\[ f(D) \ y = x \]

Define \( \frac{1}{f(D)} \) such that

\[ f(D) \left( \frac{1}{f(D)} \right) x = x \]

\[ \text{...(2)} \]

Here \( f(D) \) is called the inverse differential operator. Hence from Eqn. (1), we obtain

\[ y = \frac{1}{f(D)} x \]

\[ \text{...(3)} \]

Since this satisfies the Eqn. (1) hence the particular integral of Eqn. (1) is given by Eqn. (3)

Thus, particular Integral (P.I.) = \( \frac{1}{f(D)} x \)

The inverse differential operator \( \frac{1}{f(D)} \) is linear.

\[ i.e., \quad \frac{1}{f(D)} \{ax_1 + bx_2\} = a \frac{1}{f(D)} x_1 + b \frac{1}{f(D)} x_2 \]

where \( a, b \) are constants and \( x_1 \) and \( x_2 \) are some functions of \( x \).
SPECIAL FORMS OF THE PARTICULAR INTEGRAL

Type 1: P.I. of the form $\frac{e^{ax}}{f(D)}$

We have the equation $f(D)\ y = e^{ax}$

Let $f(D) = D^2 + a_1 D + a_2$

We have $D\ (e^{ax}) = a\ e^{ax}$, $D^2\ (e^{ax}) = a^2\ e^{ax}$ and so on.

$\therefore \ f(D)\ e^{ax} = (D^2 + a_1 D + a_2)\ e^{ax}$

$\quad = a^2\ e^{ax} + a_1\ a e^{ax} + a_2\ e^{ax}$

$\quad = (a^2 + a_1\ a + a_2)\ e^{ax} = f(a)\ e^{ax}$

Thus $f(b)\ e^{ax} = f(a)\ e^{ax}$

Operating with $\frac{1}{f(D)}$ on both sides

We get,

$e^{ax} = f(a)\cdot\frac{1}{f(D)}\cdot e^{ax}$

or

P.I. $= \frac{1}{f(D)}\ e^{ax} = \frac{e^{ax}}{f(D)}$

In particular if $f(D) = D - a$, then using the general formula.

We get,

$\frac{1}{D-a}\ e^{ax} = \frac{e^{ax}}{(D-a)\phi(D)} = \frac{1}{D-a}\phi(a)$

\[i.e., \quad \frac{e^{ax}}{f(D)} = \frac{1}{\phi(a)}\ e^{ax}\int e^{ax} \ dx = \frac{1}{\phi(a)}\cdot x\ e^{ax} \quad \ldots(1)\]

$\therefore \ f'(a) = 0 + \phi(a)$

or

$f(a) = \phi(a)$

Thus, Eqn. (1) becomes

$\frac{e^{ax}}{f(D)} = x\cdot\frac{e^{ax}}{f''(D)}$

where

$f(a) = 0$

and

$f'(a) \neq 0$

This result can be extended further also if

$f'(a) = 0, \ \frac{e^{ax}}{f(D)} = x^2\cdot\frac{e^{ax}}{f''(a)}$ and so on.

Type 2: P.I. of the form $\frac{\sin ax}{f(D)}$, $\frac{\cos ax}{f(D)}$

We have $D\ (\sin ax) = a\ \cos ax$
\[ D^2 (\sin ax) = -a^2 \sin ax \]
\[ D^3 (\sin ax) = -a^3 \cos ax \]
\[ D^4 (\sin ax) = a^4 \sin ax \]
\[ = (-a^2)^2 \sin ax \] and so on.

Therefore, if \( f(D^2) \) is a rational integral function of \( D^2 \) then \( f(D^2) \sin ax = f(-a^2) \sin ax \).

Hence \[ \frac{1}{f(D^2)} \{ f(D^2) \sin ax \} = \frac{1}{f(D^2)} f(-a^2) \sin ax \]

i.e., \[ \sin ax = f(-a^2) \frac{1}{f(D^2)} \sin ax \]

i.e., \[ \frac{1}{f(D^2)} \sin ax = \frac{\sin ax}{f(-a^2)} \]

Provided \( f(-a^2) \neq 0 \) ... (1)

Similarly, we can prove that \[ \frac{1}{f(D^2)} \cos ax = \frac{\cos ax}{f(-a^2)} \]

if \( f(-a^2) \neq 0 \)

In general, \[ \frac{1}{f(D^2)} \cos ax = \frac{\cos ax}{f(-a^2)} \]

if \( f(-a^2) \neq 0 \) ... (2)

\[ \frac{1}{f(D^2)} \sin (ax + b) = \frac{1}{f(-a^2)} \sin (ax + b) \]

and \[ \frac{1}{f(D^2)} \cos (ax + b) = \frac{1}{f(-a^2)} \cos (ax + b) \]

These formula can be easily remembered as follows.

\[ \frac{1}{D^2 + a^2} \sin ax = \frac{x}{2} \int \sin ax \, dx = -\frac{x}{2a} \cos ax \]

\[ \frac{1}{D^2 + a^2} \cos ax = \frac{x}{2} \int \cos ax \, dx = \frac{x}{2a} \sin ax. \]

**Type 3:** P.I. of the form \( \frac{\phi(x)}{f(D)} \) where \( \phi(x) \) is a polynomial in \( x \), we seeking the polynomial

Eqn. as the particular solution of \( f(D)y = \phi(x) \)

where \( \phi(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n \)

Hence P.I. is found by divisor. By writing \( \phi(x) \) in descending powers of \( x \) and \( f(D) \) in ascending powers of \( D \). The division get completed without any remainder. The quotient so obtained in the process of division will be particular integral.
Type 4: P.I. of the form $\frac{e^{ax} V}{f(D)}$ where $V$ is a function of $x$.

We shall prove that $\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D + a)} V$.

Consider

$$D (e^{ax} V) = e^{ax} DV + Va e^{ax}$$

$$= e^{ax} (D + a) V$$

and

$$D^2 (e^{ax} V) = e^{ax} D^2 V + a e^{ax} DV + a^2 e^{ax} V + a e^{ax} DV$$

$$= e^{ax} (D^2 V + 2a DV + a^2 V)$$

$$= e^{ax} (D + a)^2 V$$

Similarly,

$$D^3 (e^{ax} V) = e^{ax} (D + a)^3 V$$ and so on.

\[ \therefore \] $f(D) e^{ax} V = e^{ax} f(D + a) V$ ... (1)

Let

$$f(D + a) V = U,$$ so that $V = \frac{1}{f(D + a)} U$.

Hence (1) reduces to

$$f(D) e^{ax} V = e^{ax} U$$

Operating both sides by $\frac{1}{f(D)}$ we get,

$$e^{ax} \frac{1}{f(D + a)} U = \frac{1}{f(D)} e^{ax} U$$

i.e.,

$$\frac{1}{f(D)} e^{ax} U = e^{ax} \frac{1}{f(D + a)} U$$

Replacing $U$ by $V$, we get the required result.

Type 5: P.I. of the form $\frac{x^V}{f(D)}, \frac{x^n V}{f(D)}$ where $V$ is a function of $x$.

By Leibniz’s theorem, we have

$$D^n (x^V) = x D^n V + n-1 D^{n-1} V$$

$$= x D^n V + \left( \frac{d}{dD} D^n \right) V$$

\[ \therefore \] $f(D) x^V = x f(D) V + f'(D) V$ ... (1)

Eqn. (1) reduces to

$$\frac{x^V}{f(D)} = \left[ x - \frac{f'(D)}{f(D)} \right] \frac{V}{f(D)}$$ ... (2)

This is formula for finding the particular integral of the functions of the $xV$. By repeated application of this formula, we can find P.I. as $x^2 V, x^3 V$ .......
Type 1

1. Solve \( \frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + 6y = e^{5x} \).

Solution. We have
\[
(D^2 - 5D + 6) y = e^{5x}
\]
A.E. is
\[
m^2 - 5m + 6 = 0
\]
\[
i.e., \quad (m - 2)(m - 3) = 0
\]
\[
\Rightarrow \quad m = 2, 3
\]
Hence the complementary function is
\[
C.F. = C_1 e^{2x} + C_2 e^{3x}
\]
Particular Integral (P.I.) is
\[
P.I. = \frac{1}{D^2 - 5D + 6} e^{5x} \quad \quad \quad \quad \quad \quad (D \rightarrow 5)
\]
\[
= \frac{1}{(5^2 - 5 \times 5 + 6)} e^{5x} = \frac{e^{5x}}{6}.
\]
\[.
\]
The general solution is given by
\[
y = C.F. + P.I.
\]
\[
= C_1 e^{2x} + C_2 e^{3x} + \frac{e^{5x}}{6}.
\]

2. Solve \( \frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} + 2y = 10e^{3x} \).

Solution. We have
\[
(D^2 - 3D + 2) y = 10 e^{3x}
\]
A.E. is
\[
m^2 - 3m + 2 = 0
\]
\[
i.e., \quad (m - 2)(m - 1) = 0
\]
\[
\Rightarrow \quad m = 2, 1
\]
C.F. = \( C_1 e^{2x} + C_2 e^{x} \)
\[
P.I. = \frac{1}{D^2 - 3D + 2} 10e^{3x} \quad \quad \quad \quad \quad \quad (D \rightarrow 3)
\]
\[
= \frac{1}{(3^2 - 3 \times 3 + 2)} 10e^{3x}
\]
\[
P.I. = \frac{10 e^{3x}}{2}
\]
\[.
\]
The general solution is
\[
y = C.F. + P.I.
\]
\[
= C_1 e^{2x} + C_2 e^{x} + \frac{10 e^{3x}}{2}.
\]
3. Solve \( \frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 4y = e^{2x} \).

**Solution.** Given equation is

\[
(D^2 - 4D + 4)y = e^{2x}
\]

A.E. is \( m^2 - 4m + 4 = 0 \)

i.e., \( (m - 2)(m - 2) = 0 \)

\[ m = 2, 2 \]

C.F. = \((C_1 + C_2) e^{2x}\)

\[
P.I. = \frac{1}{D^2 - 4D + 4} e^{2x} \quad (D = 2)
\]

\[
= \frac{1}{2^2 - 4(2) + 4} e^{2x} \quad (Dr = 0)
\]

Differentiate the denominator and multiply \( x \)

\[
= x \cdot \frac{1}{2D - 4} e^{2x} \quad (D \rightarrow 2)
\]

\[
= x \cdot \frac{1}{2(2) - 4} e^{2x} \quad (Dr = 0)
\]

Again differentiate denominator and multiply \( x \)

\[
= x^2 \cdot \frac{1}{2} e^{2x}
\]

\[
P.I. = \frac{x^2 e^{2x}}{2}
\]

\[
y = C.F. + P.I. = (C_1 + C_2 x) e^{2x} + \frac{x^2 e^{2x}}{2}.
\]
Type2:

1. Solve \((D^3 + D^2 - D - 1)\ y = \cos\ 2x.\)

Solution. The A.E. is

\[
m^3 + m^2 - m - 1 = 0
\]
i.e., \(m(m + 1)^2 - (m + 1) = 0\)

\((m + 1)(m^2 - 1) = 0\)

\(m = -1, \ m^2 = 1\)
\(m = -1, \ m = \pm 1\)
\(m = -1, -1, 1\)

\[
C.F. = C_1 e^x + (C_2 + C_3 x) e^{-x}
\]

P.I. = \[
\frac{1}{D^3 + D^2 - D - 1} \cos 2x
\]
\[
= \frac{1}{(D + 1)(D^2 - 1)} \cos 2x
\]
\[
= \frac{1}{(D + 1)(-2^2 - 1)} \cos 2x
\]
\[
= \frac{-1}{5} \frac{1}{D + 1} \cos 2x
\]
\[
= \frac{-1}{5} \cos 2x \times \frac{D - 1}{D + 1}
\]
\[
= \frac{-1}{5} \frac{(D - 1) \cos 2x}{D^2 - 1}
\]
\[
= \frac{-1}{5} \left[ \frac{-2 \sin 2x - \cos 2x}{-2^2 - 1} \right]
\]
\[
= \frac{-1}{25} (2 \sin 2x + \cos 2x)
\]

\[
\therefore \text{ The general solution is}
\]

\[
y = C.F. + P.I.
\]

\[
= C_1 e^x + (C_2 + C_3 x) e^{-x} - \frac{1}{25} (2 \sin 2x + \cos 2x).
\]
2. Solve \((D^2 + D + 1) y = \sin 2x\).

Solution. The A.E. is
\[
m^2 + m + 1 = 0
\]
\[i.e., \quad m = \frac{-1 \pm \sqrt{1 - 4}}{2} = \frac{-1 \pm \sqrt{3} i}{2}\]

Hence the C.F. is
\[
C.F. = e^{-\frac{x}{2}} \left[ C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x \right]
\]
\[
P.I. = \frac{1}{D^2 + D + 1} \sin 2x
\]
\[= \frac{1}{-2^2 + D + 1} \sin 2x
\]
\[= \frac{1}{D - 3} \sin 2x
\]

Multiplying and dividing by \((D + 3)\)
\[
= \frac{(D + 3) \sin 2x}{D^2 - 9}
\]
\[= \frac{(D + 3) \sin 2x}{-2^2 - 9} - \frac{1}{13} (2 \cos 2x + 3 \sin 2x)
\]
\[
\therefore \quad y = C.F. + P.I. = e^{-\frac{x}{2}} \left[ C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x \right] - \frac{1}{3} (2 \cos 2x + 3 \sin 2x).
\]

3. Solve \((D^2 + 5D + 6) y = \cos x + e^{2x}\).

Solution. The A.E. is
\[
m^2 + 5m + 6 = 0
\]
\[i.e., \quad (m + 2)(m + 3) = 0
\]
\[
m = -2, -3
\]
\[
C.F. = C_1 e^{-2x} + C_2 e^{-3x}
\]
\[
P.I. = \frac{1}{D^2 + 5D + 6} \cdot [\cos x + e^{2x}]
\]
\[= \frac{\cos x}{D^2 + 5D + 6} + \frac{e^{2x}}{D^2 + 5D + 6}
\]
\[= P.I. + P.I.
\]
\[
P.I._1 = \frac{\cos x}{D^2 + 5D + 6}
\]
\[= \frac{\cos x}{-1^2 + 5D + 6} = \frac{\cos x}{5D + 5}
\]
\[
D^2 = -1^2
\]
\[
\begin{align*}
\cos x (D-1) & = \frac{1}{5} \cos x (D-1) \\
& = \frac{1}{5} \frac{(D-1) \cos x}{D^2 - 1} \\
& = \frac{1}{5} \frac{-\sin x - \cos x}{-1^2 - 1} \\
& = \frac{1}{5} \frac{\sin x + \cos x}{-2} \\
& = \frac{1}{10} (\sin x + \cos x) \\
\end{align*}
\]

P.I. \( _2 \) = \frac{e^{-2x}}{D^2 + 5D + 6} \quad (D \to -2)

\[
\begin{align*}
& = \frac{e^{-2x}}{(-2)^2 + 5 \times -2 + 6} \\
& = \frac{e^{-2x}}{4 - 10 + 6} \\
& = \frac{e^{-2x}}{0} \\
& = \infty \\
\end{align*}
\]

Differential and multiply ‘x’

\[
\begin{align*}
x e^{-2x} & = \frac{x e^{-2x}}{2D + 5} \\
& = \frac{x e^{-2x}}{2(-2) + 5} \\
& = \frac{x e^{-2x}}{1} \\
& = x e^{-2x} \\
\end{align*}
\]

P.I. = \( \frac{1}{10} (\sin x + \cos x) + x e^{-2x} \)

\[
\begin{align*}
\therefore \text{The general solution is} \\
y & = \text{C.F.} + \text{P.I.} \\
y & = C_1 e^{-2x} + C_2 e^{-3x} + \frac{1}{10} (\sin x + \cos x) + x e^{-2x}.
\end{align*}
\]
Type 3

1. Solve \( y'' + 3y' + 2y = 12x^2 \).

Solution. We have \((D^2 + 3D + 2) y = 12x^2\)
A.E. is \( m^2 + 3m + 2 = 0 \)
i.e., \((m + 1)(m + 2) = 0 \)
\( \Rightarrow \quad m = -1, -2 \)
C.F. \( = C_1 e^{-x} + C_2 e^{-2x} \)
P.I. \( = \frac{12x^2}{D^2 + 3D + 2} \)

We need to divide for obtaining the P.I.

\[
\begin{array}{c|c|c|c}
2 + 3D + D^2 & 6x^2 - 18x + 21 \\
\hline
12x^2 & 12x^2 \\
12x^2 + 36x + 12 & -36x - 12 \\
-36x - 54 & -36x - 54 \\
42 & 42 \\
0 & 0 \\
\end{array}
\]

Note:
\( 3D(6x^2) = 36x \)
\( D^2(6x^2) = 12 \)

Hence, P.I. \( = 6x^2 - 18x + 21 \)

\( \therefore \) The general solution is
\( y = \text{C.F.} + \text{P.I.} \)
\( y = C_1 e^{-x} + C_2 e^{-2x} + 6x^2 - 18x + 21 \).

2. Solve \( \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + y = 2x + x^2 \).

Solution. We have \((D^2 + 2D + 1) y = 2x + x^2\)
A.E. is \( m^2 + 2m + 1 = 0 \)
i.e., \((m + 1)^2 = 0 \)
i.e., \((m + 1)(m + 1) = 0 \)
\( \Rightarrow \quad m = -1, -1 \)
C.F. \( = (C_1 + C_2 x) e^{-x} \)
P.I. \( = \frac{2x + x^2}{D^2 + 2D + 1} = \frac{x^2 + 2x}{1 + 2D + D^2} \)
\[
1 + 2D + D^2 \ \frac{x^2 - 2x + 2}{x^2 + 2x} \ \frac{x^2 + 4x + 2}{-2x - 2} \ \frac{-2x - 4}{2} \ \frac{2}{2} \ \frac{0}{0}
\]

\[
\therefore \quad \text{P.I.} = x^2 - 2x + 2
\]

\[
\therefore \quad y = \text{C.F.} + \text{P.I.}
\]

\[
= (C_1 + C_2 \cdot x) \ e^{-x} + (x^2 - 2x + 2).
\]

**Type 4**

1. Solve \( \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} - 3y = e^x \cos x \).

**Solution.** We have

\[
(D^2 + 2D - 3) y = e^x \cos x
\]

A.E. is

\[
m^2 + 2m - 3 = 0
\]

i.e.,

\[
(m + 3)(m - 1) = 0
\]

i.e.,

\[
m = -3, 1
\]

C.F. = \( C_1 \ e^{-3x} + C_2 \ e^x \)

P.I. = \( \frac{1}{D^2 + 2D - 3} \ e^x \cos x \)

Taking \( e^x \) outside the operator and changing \( D \) to \( D + 1 \)

\[
= e^x \ \frac{1}{(D+1)^2 + 2(D+1) - 3} \ \cos x
\]

\[
= e^x \ \frac{1}{D^2 + 4D} \ \cos x \quad (D^2 \rightarrow -1^2)
\]
\[ = e^x \frac{1}{-1 + 4D} \cos x \]

\[ = e^x \left[ \cos x \frac{4D + 1}{4D - 1} \right] \]

\[ = e^x \left[ \frac{-4 \sin x + \cos x}{16 D^2 - 1} \right] \]

\[ = e^x \left[ \frac{-4 \sin x + \cos x}{-17} \right] \]

\[ = \frac{e^x}{17} (4 \sin x - \cos x) \]

**: y = C.F. + P.I.

\[ y = C_1 e^{-3x} + C_2 e^x + \frac{e^x}{17} (4 \sin x - \cos x). \]

2. Solve \((D^3 + 1) y = 5e^x x^2\).

Solution. A.E. is

\[ m^3 + 1 = 0 \]

i.e., \((m + 1) (m^2 - m + 1) = 0\)

\((m + 1) = 0, \ m^2 - m + 1 = 0\)

\[ m = -1 \]

\[ m = \frac{1 \pm \sqrt{3}i}{2} \]

C.F. \[ = C_1 e^{-x} + e^x \left( C_2 \cos \frac{\sqrt{3}}{2} x + C_3 \sin \frac{\sqrt{3}}{2} x \right) \]

P.I. \[ = \frac{1}{D^3 + 1} 5e^x x^2 \]

Taking \(e^x\) outside the operator and changing \(D\) to \(D + 1\)

\[ = e^x \frac{1}{(D + 1)^3 + 1} \cdot 5x^2 \]

\[ = e^x \frac{5x^2}{D^3 + 3D^2 + 3D + 2} \]
\[ y = \frac{5e^x}{2} \left( x^2 - 3x + \frac{3}{2} \right) \]

(For a convenient division we have multiplied and divided by 2)

\[
\begin{array}{c}
2 + 3D + 3D^2 + D^3 \\
2x^2 \\
2x^2 + 6x + 6 \\
-6x - 6 \\
-6x - 9 \\
3 \\
3 \\
0 \\
\end{array}
\]

\[ \therefore \quad \text{P.I.} = \left( x^2 - 3x + \frac{3}{2} \right) \frac{5e^x}{2} \]

\[ = \frac{5e^x}{4} (2x^2 - 6x + 3) \]

\[ y = \text{C.F.} + \text{P.I.} \]

\[ = C_1 e^{-x} + e^{x} \left\{ C_2 \cos \frac{\sqrt{3}}{2} x + C_3 \sin \frac{\sqrt{3}}{2} x \right\} + \frac{5e^x}{4} (2x^2 - 6x + 3). \]
Type 5

1. Solve \( \frac{d^2y}{dx^2} + 4y = x \sin x \).

Solution. We have

\[
(D^2 + 4) y = x \sin x
\]

A.E. is

\[
m^2 + 4 = 0
\]

\[
m^2 = -4
\]

\[
m = \pm 2i
\]

C.F. = \( C_1 \cos 2x + C_2 \sin 2x \)

P.I. = \( \frac{1}{D^2 + 4} x \sin x \)

Let us use

\[
\frac{xV}{f(D)} = \left[ x - \frac{f'(D)}{f(D)} \right] \frac{V}{f(D)}
\]

\[
\frac{x \sin x}{D^2 + 4} = \left[ x - \frac{2D}{D^2 + 4} \right] \frac{\sin x}{D^2 + 4}
\]

\[
= \frac{x \sin x}{D^2 + 4} - \frac{2D(\sin x)}{(D^2 + 4)^2}
\]

\[
(D^2 \rightarrow -1^2)
\]

\[
(D^2 \rightarrow -1^2)
\]

\[
= \frac{x \sin \frac{x}{3}}{3} - \frac{2 \cos x}{3^2}
\]

\[
= \frac{x \sin \frac{x}{3}}{3} - \frac{2 \cos x}{9}
\]

P.I. = \( \frac{1}{9} (3x \sin x - 2 \cos x) \)

\[y = \text{C.F.} + \text{P.I.}\]

\[= C_1 \cos 2x + C_2 \sin 2x + \frac{1}{9} (3x \sin x - 2 \cos x)\]

2. Solve \( (D^2 + 2D + 1) y = x \cos x \).

Solution. A.E. is

\[
m^2 + 2m + 1 = 0
\]

\[
(m + 1)^2 = 0
\]

\[
m = -1, -1
\]

C.F. = \( (C_1 + C_2 x) e^{-x} \)

P.I. = \( \frac{x \cos x}{D^2 + 2D + 1} \).
Let us have

\[ \frac{xV}{f(D)} = \left[ x - \frac{f'(D)}{f(D)} \right] \frac{V}{f(D)} \]

\[ = \left[ x - \frac{2D + 2}{D^2 + 2D + 1} \right] \frac{\cos x}{D^2 + 2D + 1} \]

\[ = \frac{x \cos x}{D^2 + 2D + 1} - \frac{(2D + 2) \cos x}{(D^2 + 2D + 1)^2} \]

\[ = P.I.1 - P.I.2 \]

**P.I.1**

\[ = \frac{x \cos x}{D^2 + 2D + 1} \]

\[ = \frac{x \cos x \times D}{2D \times D} \]

\[ = \frac{-x \sin x}{2D^2} \quad (D^2 \rightarrow -1^2) \]

**P.I.1**

\[ = \frac{x}{2} \sin x \]

**P.I.2**

\[ = \frac{(2D + 2) \cos x}{(D^2 + 2D + 1)^2} \]

\[ = \frac{-2 \sin x + 2 \cos x}{(2D)^2} \]

\[ = \frac{-2 \sin x + 2 \cos x}{4D^2} \]

\[ = \frac{2 \sin x - 2 \cos x}{4} \]

\[ = \frac{1}{2} (\sin x - \cos x) \]

**P.I.**

\[ = \frac{1}{2} x \sin x - \frac{1}{2} (\sin x - \cos x) \]

\[ = \frac{1}{2} (x \sin x - \sin x + \cos x) \]

\[ y = C.F. + P.I. \]

\[ y = (C_1 + C_2 x) e^{-x} + \frac{1}{2} (x \sin x - \sin x + \cos x). \]
SOLUTION OF SIMULTANEOUS DIFFERENTIAL EQUATIONS

Let us suppose that x and y are functions of an independent variable ‘t’ connected by a system of first order equation with \( D = \frac{d}{dt} \)

\[
\begin{align*}
  f_1(D) \ x + f_2(D) \ y &= \phi_1(t) \\
  g_1(D) \ x + g_2(D) \ y &= \phi_2(t)
\end{align*}
\]

By solving a system of linear algebraic equations in cancelling either of the dependent variables (x or y) operating (1) with \( g_1(D) \) and (2) with \( f_1(D) \), x cancels out by subtraction. We obtain a second order differential equation in y. Which can be solved x can be obtained independently by cancelling y or by substituting the obtained y (t) in a suitable equation.

Let us suppose that x and y are functions of an independent variable ‘t’ connected by a system of first order equation with \( D = \frac{d}{dt} \)

\[
\begin{align*}
  f_1(D) \ x + f_2(D) \ y &= \phi_1(t) \\
  g_1(D) \ x + g_2(D) \ y &= \phi_2(t)
\end{align*}
\]

By solving a system of linear algebraic equations in cancelling either of the dependent variables (x or y) operating (1) with \( g_1(D) \) and (2) with \( f_1(D) \), x cancels out by subtraction. We obtain a second order differential equation in y. Which can be solved x can be obtained independently by cancelling y or by substituting the obtained y (t) in a suitable equation.
1. Solve \( \frac{dx}{dt} - 7x + y = 0, \quad \frac{dx}{dt} - 2x - 5y = 0. \)

**Solution.** Taking \( D = \frac{d}{dt} \), we have the system of equations

\[
(D - 7) x + y = 0 \quad \text{...}(1)
\]
\[
- 2x + (D - 5) y = 0 \quad \text{...}(2)
\]

Multiply (1) by 2 and operate (2) by \((D - 7)\)

\[
i.e., \quad 2 (D - 7) x + 2y = 0
\]
\[
- 2 (D - 7) x + (D - 5) (D - 7) y = 0
\]

Adding \([D^2 - 12D + 37] y = 0\) or

\[\begin{align*}
(D^2 - 12D + 37) y &= 0 \\
(m - 6)^2 + 1 &= 0
\end{align*}\]

A.E. is \( m^2 - 12m + 37 = 0 \)

\[
(m - 6)^2 + 1 = 0
\]

\[\Rightarrow \quad m - 6 = \pm i
\]

\[\Rightarrow \quad m = 6 \pm i
\]

Thus

\[y = e^{6t} (C_1 \cos t + C_2 \sin t) \quad \text{...}(3)
\]

By considering \( \frac{dy}{dt} - 2x - 5y = 0 \), we get

\[x = \frac{1}{2} \left( \frac{dy}{dt} - 5y \right)
\]

\[\therefore \quad x = \frac{1}{2} \left[ e^{6t} (C_1 \cos t + C_2 \sin t) - 5e^{6t} (C_1 \cos t + C_2 \sin t) \right]
\]

\[= \frac{1}{2} \left[ e^{6t} (- C_1 \sin t + C_2 \cos t) + 6e^{6t} (C_1 \cos t + C_2 \sin t) - 5e^{6t} (C_1 \cos t + C_2 \sin t) \right]
\]

\[\Rightarrow \quad x = \frac{1}{2} \left( e^{6t} (- C_1 \sin t + C_2 \cos t) + e^{6t} (C_1 \cos t + C_2 \sin t) \right)
\]

Thus

\[x = \frac{1}{2} \left( (C_1 + C_2) e^{6t} \cos t + (C_2 - C_1) e^{6t} \sin t \right) \quad \text{...}(4)
\]

(3) and (4) represents the complete solution of the given system of equations.
2. Solve: \( \frac{dx}{dt} = 2x - 3y, \frac{dy}{dt} = y - 2x \) given \( x(0) = 8 \) and \( y(0) = 3 \).

**Solution.** Taking \( D = \frac{d}{dt} \) we have the system of equations,

\[
Dx = 2x - 3y; \quad Dy = y - 2x
\]

i.e.,
\[
(D - 2) x + 3y = 0 \quad \ldots(1)
\]
\[
2x + (D - 1) y = 0 \quad \ldots(2)
\]

Multiplying (1) by 2 and (2) by \((D - 2)\), we get
\[
2(D - 2) x + 6y = 0 \quad \ldots(3)
\]
\[
2(D - 2) x + (D - 1)(D - 2) y = 0 \quad \ldots(4)
\]

Subtracting, we get \((D^2 - 3D - 4) y = 0\)

A.E. is
\[
m^2 - 3m - 4 = 0
\]

or
\[
(m - 4)(m + 1) = 0 \quad \Rightarrow \quad m = 4, -1
\]

\[
\therefore \quad y = C_1 e^{4t} + C_2 e^{-t} \quad \ldots(3)
\]

By considering \( \frac{dy}{dt} = y - 2x \), we get

\[
x = \frac{1}{2} \left[ y - \frac{dy}{dt} \right]
\]

i.e.,
\[
x = \frac{1}{2} \left[ C_1 e^{4t} + C_2 e^{-t} - \left( 4C_1 e^{4t} - C_2 e^{-t} \right) \right]
\]
\[
= \frac{1}{2} \left( -3C_1 e^{4t} + 2C_2 e^{-t} \right) \quad \ldots(4)
\]

We have conditions \( x = 8, y = 3 \) at \( t = 0 \)

Hence (3) and (4) become \( C_1 + C_2 = 3 \) and \( -\frac{3C_1}{2} + C_2 = 8 \).

Solving these equations, we get \( C_2 = 5, C_1 = -2 \)

Thus
\[
x = 3e^{4t} + 5e^{-t}
\]
\[
y = -2e^{4t} + 5e^{-t} \] is the required solution.

3. Solve: \( \frac{dx}{dt} - 2y = \cos 2t, \frac{dy}{dt} + 2x = \sin 2t \) given that \( x = 1, y = 0 \) at \( t = 0 \).

**Solution.** Taking \( D = \frac{d}{dt} \) we have the system of equations

\[
Dx - 2y = \cos 2t \quad \ldots(1)
\]
\[
2x + Dy = \sin 2t \quad \ldots(2)
\]

Multiplying (1) by \( D \) and (2) by 2, we have
\[ D^2x - 2Dy = D(\cos 2t) = -2 \sin 2t \]
\[ 4x + 2Dy = 2 \sin 2t \]

Adding, we get \((D^2 + 4) x = 0\)

A.E. is 
\[ m^2 + 4 = 0 \quad \Rightarrow \quad m = \pm 2i \]

\[ x = C_1 \cos 2t + C_2 \sin 2t \quad \ldots(3) \]

By considering 
\[ \frac{dx}{dt} - 2y = \cos 2t, \] we get

\[ y = \frac{1}{2} \left[ \frac{dx}{dt} - \cos 2t \right] \]

\[ i.e., \quad y = \frac{1}{2} \left[ \frac{d}{dt}(C_1 \cos 2t + C_2 \sin 2t) - \cos 2t \right] \]

\[ = \frac{1}{2} \left[ -2C_1 \sin 2t + 2C_2 \cos 2t - \cos 2t \right] \]

\[ y = -C_1 \sin 2t + \left( C_2 - \frac{1}{2} \right) \cos 2t \quad \ldots(4) \]

Equation (3) and (4) represents the general solution

Applying the given conditions \(x = 1\) at \(t = 0\)

Hence (3) becomes, 
\[ 1 = C_1 + 0 \quad \Rightarrow \quad C_1 = 1 \]

\[ y = 0 \text{ at } t = 0 \]

Hence (4) becomes, 
\[ 0 = 0 + \left( C_2 - \frac{1}{2} \right) \quad \Rightarrow \quad C_2 = \frac{1}{2} \]

Substituting these values in (3) and (4), we get

\[ x = \cos 2t + \frac{1}{2} \sin 2t \]

\[ y = -\sin 2t \]

Which is the required solution.
UNIT III

DIFFERENTIAL EQUATIONS – III

METHOD OF VARIATION OF PARAMETERS

Consider a linear differential equation of second order
\[ \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = \phi(x) \] \hspace{1cm} ...(1)

where \( a_1, a_2 \) are functions of ‘\( x \)’. If the complimentary function of this equation is known then we can find the particular integral by using the method known as the method of variation of parameters.

Suppose the complimentary function of the Eqn. (1) is
\[ \text{C.F.} = C_1 y_1 + C_2 y_2 \]
where \( C_1 \) and \( C_2 \) are constants and \( y_1 \) and \( y_2 \) are the complementary solutions of Eqn. (1)

The Eqn. (1) implies that
\[ y''_1 + a_1 y'_1 + a_2 y_1 = 0 \] \hspace{1cm} ...(2)
\[ y''_2 + a_1 y'_2 + a_2 y_2 = 0 \] \hspace{1cm} ...(3)

We replace the arbitrary constants \( C_1, C_2 \) present in C.F. by functions of \( x \), say \( A, B \) respectively,
\[ \therefore \quad y = Ay_1 + By_2 \] \hspace{1cm} ...(4)

is the complete solution of the given equation.

The procedure to determine \( A \) and \( B \) is as follows.

From Eqn. (4)
\[ y' = (Ay'_1 + By'_2) + (A'y_1 + B'y_2) \] \hspace{1cm} ...(5)

We shall choose \( A \) and \( B \) such that
\[ A'y_1 + B'y_2 = 0 \] \hspace{1cm} ...(6)

Thus Eqn. (5) becomes \[ y'_1 = Ay'_1 + By'_2 \] \hspace{1cm} ...(7)

Differentiating Eqn. (7) w.r.t. ‘\( x \)’ again, we have
\[ y'' = (Ay''_1 + Ay''_2) + (A'y'_1 + B'y'_2) \] \hspace{1cm} ...(8)

Thus, Eqn. (1) as a consequence of (4), (7) and (8) becomes
\[ A'y'_1 + B'y'_2 = \phi(x) \] \hspace{1cm} ...(9)
Let us consider equations (6) and (9) for solving
\[ A'y_1 + B'y_2 = 0 \]  
\[ A'y'_1 + B'y'_2 = \phi(x) \]  
...(6)  
...(9)

Solving \( A' \) and \( B' \) by cross multiplication, we get
\[ A' = \frac{-y_2 \phi(x)}{W}, \quad B' = \frac{y_1 \phi(x)}{W} \]  
...(10)

Find \( A \) and \( B \)

Integrating,
\[ A = -\int \frac{y_2 \phi(x)}{W} \, dx + k_1 \]
\[ B = \int \frac{y_1 \phi(x)}{W} \, dx + k_2 \]

where \( W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1 y'_2 - y_2 y'_1 \)

Substituting the expressions of \( A \) and \( B \)
\[ y = Ay_1 + By_2 \] is the complete solution.

1. Solve by the method of variation of parameters

\[ \frac{d^2 y}{dx^2} + y = \cosec x. \]

Solution. We have
\[ (D^2 + 1) y = \cosec x \]

A.E. is
\[ m^2 + 1 = 0 \quad \Rightarrow \quad m^2 = -1 \quad \Rightarrow \quad m = \pm i \]

Hence the C.F. is given by
\[ y_c = C_1 \cos x + C_2 \sin x \]  
\[ y = A \cos x + B \sin x \]  
...(1)  
...(2)

be the complete solution of the given equation where \( A \) and \( B \) are to be found.

The general solution is \[ y = Ay_1 + By_2 \]

We have
\[ y_1 = \cos x \] and \[ y_2 = \sin x \]
\[ y'_1 = -\sin x \] and \[ y'_2 = \cos x \]
\[ W = y_1 y'_2 - y_2 y'_1 \]
\[ = \cos x \cdot \cos x + \sin x \cdot \sin x = \cos^2 x + \sin^2 x = 1 \]
\[ A' = \frac{-y_2 \phi(x)}{W}, \quad B' = \frac{y_1 \phi(x)}{W} \]
\[ = \frac{-\sin x \cdot \csc x}{1}, \quad B' = \frac{\cos x \cdot \csc x}{1} \]
\[ A' = -1, \quad B' = \cot x \]

\[ A = \int (-1) \, dx + C_1, \text{ i.e., } A = -x + C_1 \]
\[ B = \int \cot x \, dx + C_2, \text{ i.e., } B = \log \sin x + C_2 \]

Hence the general solution of the given Eqn. (2) is
\[ y = C_1 \cos x + C_2 \sin x - x \cos x + \sin x \log \sin x. \]

2. Solve by the method of variation of parameters

\[ \frac{d^2 y}{dx^2} + 4y = 4 \tan 2x. \]

Solution. We have
\[ (D^2 + 4) \, y = 4 \tan 2x \]
A.E. is
\[ m^2 + 4 = 0 \]
where \( \phi(x) = 4 \tan 2x. \)
\[ i.e., \quad m = \pm 2i \]

Hence the complementary function is given by
\[ y_c = C_1 \cos 2x + C_2 \sin 2x \]
\[ y = A \cos 2x + B \sin 2x \]
be the complete solution of the given equation where \( A \) and \( B \) are to be found

We have
\[ y_1 = \cos 2x \quad \text{and} \quad y_2 = \sin 2x \]
\[ y_1' = -2 \sin 2x \quad \text{and} \quad y_2' = 2 \cos 2x \]

Then
\[ W = y_1y_2' - y_2y_1' \]
\[ = \cos 2x \cdot 2 \cos 2x + 2 \sin 2x \cdot \sin 2x \]
\[ = 2(\cos^2 2x + \sin^2 2x) \]
\[ = 2 \]

Also,
\[ \phi(x) = 4 \tan 2x \]
\[ A' = \frac{-y_2 \phi(x)}{W} \quad \text{and} \quad B' = \frac{y_1 \phi(x)}{W} \]
\[ A' = \frac{-\sin 2x \cdot 4 \tan 2x}{2}, \quad B' = \frac{-\cos 2x \cdot 4 \tan 2x}{2} \]
\[ A' = \frac{-2 \sin^2 2x}{\cos 2x}, \quad B' = 2 \sin 2x \]

On integrating, we get
\[ A = -2 \int \frac{\sin^2 2x}{\cos 2x} \, dx, \quad B = 2 \int \sin 2x \, dx \]
\[ = -2 \int \frac{1 - \cos^2 2x}{\cos 2x} \, dx \]
\[ = -2 \int \left\{ \sec 2x - \cos 2x \right\} \, dx \]
\[ = -2 \left\{ \frac{1}{2} \log (\sec 2x + \tan 2x) - \frac{1}{2} \sin 2x \right\} \]
\[ A = -\log (\sec 2x + \tan 2x) + \sin 2x + C_1 \]
\[ B = 2 \int \sin 2x \, dx \]
\[ = \frac{2 (-\cos 2x)}{2} + C_2 \]
\[ B = -\cos 2x + C_2 \]

Substituting these values of \( A \) and \( B \) in Eqn. (1), we get
\[ y = C_1 \cos 2x + C_2 \sin 2x - \cos 2x \log (\sec 2x + \tan 2x) \]
which is the required general solution.

**SOLUTION OF CAUCHY’S HOMOGENEOUS LINEAR EQUATION AND LEGENDRE’S LINEAR EQUATION**

A linear differential equation of the form
\[ x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = \phi(x) \]...(1)

Where \( a_1, a_2, a_3 \ldots a_n \) are constants and \( \phi(x) \) is a function of \( x \) is called a homogeneous linear differential equation of order \( n \).

The equation can be transformed into an equation with constant coefficients by changing the independent variable \( x \) to \( z \) by using the substitution \( x = e^z \) or \( z = \log x \)

Now
\[ z = \log x \Rightarrow \frac{dz}{dx} = \frac{1}{x} \]
Consider
\[
\frac{dy}{dx} = \frac{dy}{dz} = \frac{1}{x} \frac{dy}{dz}
\]
\[
\therefore \quad \frac{dy}{dx} = \frac{dy}{dz} = Dy
\]
where \(D = \frac{d}{dz}\).

Differentiating w.r.t. ‘x’ we get,
\[
\frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{1}{x} \frac{dy}{dx} = \frac{d^2y}{dz^2} \frac{dz}{dx}
\]

\[i.e., \quad \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} \cdot \frac{1}{x} \frac{dy}{dx}
\]

\[\quad = \frac{1}{x} \frac{d^2y}{dz^2} \cdot \frac{1}{x} \frac{dy}{dz}
\]

\[i.e., \quad \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} \frac{dy}{dz}
\]

\[i.e., \quad \frac{d^3y}{dx^3} = (D^2 - D) \frac{dy}{dz} = D \left(D - 1\right) \frac{dy}{dz}
\]

Similarly,
\[
\frac{d^n y}{dx^n} = D \left(D - 1\right) \left(D - 2\right) \cdots \left(D - n + 1\right) \frac{dy}{dz}
\]

Substituting these values of \(\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^n y}{dx^n}\) in Eqn. (1), it reduces to a linear differential equation with constant coefficient can be solved by the method used earlier.

Also, an equation of the form,
\[
(ax + b)^n \cdot \frac{d^n y}{dx^n} + a_1 (ax + b)^{n-1} \cdot \frac{d^{n-1} y}{dx^{n-1}} + \ldots \text{any} = (x)
\]

\[\ldots (2)
\]
where \(a_1, a_2, \ldots, a_n\) are constants and \(\phi (x)\) is a function of \(x\) is called a homogeneous linear differential equation of order \(n\). It is also called “Legendre’s linear differential equation”.

This equation can be reduced to a linear differential equation with constant coefficients by using the substitution.
\[
ax + b = e^z \text{ or } z = \log (ax + b)
\]

As above we can prove that
\[
(ax + b) \cdot \frac{dy}{dx} = a \cdot Dy
\]
\[(ax + b)^2 \cdot \frac{d^2 y}{dx^2} = a^2 \ D \ (D - 1) \ y\]

\[\text{..........................}\]

\[(ax + b)^n \cdot \frac{d^n y}{dx^n} = a^n \ D \ (D - 1)(D - 2) \ldots \ (D - n + 1) \ y\]

The reduced equation can be solved by using the methods of the previous section.

**PROBLEMS:**

1. **Solve** \(x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^4\).

**Solution.** The given equation is

\[x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^4\]  \quad ...(1)

Substitute \(x = e^z\) or \(z = \log x\)

So that \(\frac{dy}{dx} = Dy, \quad x^2 \frac{d^2 y}{dx^2} = D (D - 1) y\)

The given equation reduces to

\[D (D - 1) y - 2Dy - 4y = (e^z)^4\]

\[D (D - 1) - 2D - 4] y = e^{4z}\]

\(i.e., \quad (D^2 - 3D - 4) y = e^{4z}\)  \quad ...(2)

which is an equation with constant coefficients

A.E. is \(m^2 - 3m - 4 = 0\)

\(i.e., \quad (m - 4)(m + 1) = 0\)

\(\therefore \quad m = 4, -1\)

C.F. is \(C_1 e^{4z} + C_2 e^{-z}\)

P.I. = \(\frac{1}{D^2 - 3D - 4} e^{4z}\)  \(D \rightarrow 4\)

\[= \frac{1}{(4)^2 - 3(4) - 4} e^{4z}\]  \(Dr = 0\)

\[= \frac{1}{2D - 3} ze^{4z}\]  \(D \rightarrow 4\)

\[= \frac{1}{(2)(4) - 3} ze^{4z}\]

\[= \frac{1}{5} ze^{4z}\)
\[
\therefore \text{ The general solution of (2) is }
\]
\[
y = C.F. + P.I.
\]
\[
y = C_1 e^{z} + C_2 e^{-z} + \frac{1}{5} z e^{4z}
\]

Substituting \( e^z = x \) or \( z = \log x \), we get
\[
y = C_1 x^4 + C_2 x^{-4} + \frac{1}{5} \log x (x^4)
\]
\[
y = C_1 x^4 + \frac{C_2}{x} + \frac{x^4}{5} \log x
\]

is the general solution of the Eqn. (1).

2. Solve \( x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 4y = (x + 1)^2 \).

**Solution.** The given equation is
\[
x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 4y = (x + 1)^2 \quad \text{...(1)}
\]

Substituting \( x = e^z \) or \( z = \log x \)

Then
\[
x \frac{dy}{dx} = Dy, \quad x^2 \frac{d^2 y}{dx^2} = D (D - 1) y
\]

\[
\therefore \text{ Eqn. (1) reduces to}
\]
\[
D (D - 1) y - 3Dy + 4y = (e^z + 1)^2
\]

i.e., \( (D^2 - 4D + 4) y = e^{2z} + 2e^z + 1 \)

which is a linear equation with constant coefficients.

A.E. is \( m^2 - 4m + 4 = 0 \)

i.e., \( (m - 2)^2 = 0 \)

\[
\therefore \quad m = 2, 2
\]

C.F. = \((C_1 + C_2 z) e^{2z}\)

P.I. = \( \frac{1}{(D - 2)^2} (e^{2z} + 2e^z + 1) \) \quad \text{...(2)}

\[
= \frac{e^{2z}}{(D - 2)^2} + \frac{2e^z}{(D - 2)^2} + \frac{e^0}{(D - 2)^2}
\]

P.I.\(_1 + \) P.I.\(_2 + \) P.I.\(_3\)

P.I.\(_1 = \frac{e^{2z}}{(D - 2)^2} \quad (D \to 2)\)

\[
= \frac{e^{2z}}{(2 - 2)^2} \quad (D = 0)
\]

\[
= \frac{ze^{2z}}{2(D - 2)} \quad (D \to 2)
\]
\[
= \frac{ze^{2z}}{2(2-2)} \quad (D^r = 0)
\]

P.I. \(_1 \quad = \frac{ze^{2z}}{2} \quad (D \rightarrow 1)\]

\[
P.I. \quad = \frac{2e^z}{(D-2)^2} \quad (D \rightarrow 0)
\]

\[
P.I. \quad = \frac{e^{0z}}{(D-2)^2} \quad = \frac{e^z}{4} = \frac{1}{4}
\]

P.I. \quad = \frac{z^2}{2} e^{2z} + 2e^z + \frac{1}{4}

The general solution of Eqn. (2) is

\[
y = C.F. + P.I.
\]

\[
y = (C_1 + C_2 x) e^{2z} + \frac{z^2 e^{2z}}{2} + 2e^z + \frac{1}{4}
\]

Substituting

\[
e^z = x \text{ or } z = \log x, \text{ we get}
\]

\[
y = (C_1 + C_2 \log x) x^2 + \frac{x^2 (\log x)^2}{2} + 2x + \frac{1}{4}
\]

is the general solution of the equation (1).

3. Solve \(x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} - 12y = x^2 \log x\).

Solution. The given Eqn. is

\[
x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} - 12y = x^2 \log x \quad \ldots(1)
\]

Substituting \(x = e^z\) or \(z = \log x\), so that

\[
x \frac{dy}{dx} = Dy, \quad \text{and} \quad x^2 \frac{d^2 y}{dx^2} = D(D-1)y
\]

Then Eqn. (1) reduces to

\[
D(D-1)y + 2Dy - 12y = e^{2z}
\]

i.e., \((D^2 + D - 12)y = ze^{2z} \quad \ldots(2)\)
which is the Linear differential equation with constant coefficients.

A.E. is \( m^2 + m - 12 = 0 \)

\[ i.e., \quad (m + 4)(m - 3) = 0 \]

\[ \therefore \quad m = -4, \ 3 \]

C.F. \( = C_1e^{4z} + C_2e^{3z} \)

P.I. \( = \frac{1}{D^2 + D - 12}ze^{2z} \)

\[ = e^{2z}\left[ \frac{z}{(D + 2)^2 + (D + 2) - 12} \right] \]

\[ = e^{2z} \left[ \frac{z}{D^2 + 5D - 6} \right] \]

\[ = \frac{1}{6}z - \frac{5}{36} \]

\[ \left[ \begin{array}{c}
-6 + 5D + D^2 \\
\hline
z \\
\hline
z - \frac{5}{6} \\
\hline
\frac{5}{6} \\
\hline
\frac{5}{6} \\
\hline
0
\end{array} \right] \]

\[ \therefore \quad \text{General solution of Eqn. (2) is} \]

\[ y = \text{C.F.} + \text{P.I.} \]

\[ y = C_1e^{-4z} + C_2e^{3z} - \frac{e^{2z}}{6} \left( z + \frac{5}{6} \right) \]

Substituting \( e^z = x \) or \( z = \log x \), we get

\[ y = C_1x^{-4} + C_2x^3 - \frac{x^2}{6} \left( \log x + \frac{5}{6} \right) \]

\[ y = \frac{C_1}{x^4} + C_2x^3 - \frac{x^2}{6} \left( \log x + \frac{5}{6} \right) \]

which is the general solution of Eqn. (1).
Series solution of differential equations:

Consider the second order ODE \( f(x) \frac{d^2 y}{dx^2} + g(x) \frac{dy}{dx} + h(x)y = 0 \), where \( f(x) \), \( g(x) \) and \( h(x) \) are functions in \( x \) and \( f(x) \neq 0 \). The series solution of above type of DE is explained as follows:

1. Assume the solution of (1) in the form \( y = \sum_{r=0}^{\infty} a_r x^r \)
2. Find the derivatives \( y' \) and \( y'' \) from the assumed solution and substitute in to the given DE which results in an infinite series with various powers of \( x \) equal to zero.
3. Now equate the coefficients of various powers of \( x \) to zero and try to obtain recurrence relation from which the constants \( a_0, a_1, a_2, \ldots \) can be determined.
4. When substituted the values of \( a_0, a_1, a_2, \ldots \) in to the assumed solution, we get the power series solution of the given DE in the form \( y = Ay_1(x) + By_2(x) \), where \( A \) and \( B \) are arbitrary constants.

In general, The above type of DE can be solved by the following two methods:

**Type – I (Frobenius Method)**

Suppose \( f(x) = 0 \) at \( x = 0 \) in the above differential equation, we assume solution in the form \( y = \sum_{r=0}^{\infty} a_r x^{k+r} \) where \( k, a_0, a_1, a_2, \ldots \) are all constants to be determined and \( a_0 \neq 0 \).

**Type – II (Power Series Method)**

Suppose \( f(x) \neq 0 \) at \( x = 0 \) in the above differential equation, we assume solution in the form \( y = \sum_{r=0}^{\infty} a_r x^r \) where \( a_0, a_1, a_2, \ldots \) are all constants to be determined. Here all the constants \( a_r \)'s will be expressed in terms of \( a_0 \) and \( a_1 \) only.

**Problems :**

1. Obtain the series solution of the equation \( \frac{d^2 y}{dx^2} + y = 0 \) \( \quad \text{----- (1)} \)

Let \( y = \sum_{r=0}^{\infty} a_r x^r \) \( \quad \text{----- (2)} \) be the series solution of (1).
Hence, \( y' = \sum_{r=0}^{\infty} a_r r x^{r-1}, \quad y'' = \sum_{r=0}^{\infty} a_r (r-1) x^{r-2}, \)

Now (1) becomes

\[
\sum_{r=2}^{\infty} a_r r(r-1) x^{r-2} + \sum_{r=1}^{\infty} a_r x^r = 0
\]

Equating the coefficients of various powers of \( x \) to zero, we get

Coefficient of \( x^2 \): \( a_0(0)(-1) = 0 \) \( \Rightarrow \) and \( a_0 \neq 0 \).

Coefficient of \( x^{-1} \): \( a_1(1)(0) = 0 \) \( \Rightarrow \) and \( a_1 \neq 0 \).

Equating the coefficient of \( x^r \) \( (r \geq 0) \)

\[
a_{r+2}(r+2)(r+1) + a_r = 0 \quad \text{or} \quad a_{r+2} = -\frac{a_r}{(r+2)(r+1)} \quad (r \geq 0) \quad --------(3)
\]

Putting \( r = 0,1,2,3,... \) in (3) we obtain,

\[
a_2 = -\frac{a_0}{2} \quad; \quad a_3 = -\frac{a_1}{6} \quad; \quad a_4 = -\frac{a_2}{24} \quad; \quad a_5 = -\frac{a_3}{120} \quad = \frac{a_1}{120};
\]

\[
a_6 = -\frac{a_4}{30} = -\frac{a_0}{720} \quad; \quad a_7 = -\frac{a_5}{42} = -\frac{a_1}{5040} \quad; \quad \text{and so on.}
\]

Substituting these values in the expanded form of (1), we get,

\[
y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots
\]

i.e., \( y = a_0 \left[ 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \right] + a_1 \left[ x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \right] \)

Hence \( y = a_0 \left[ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right] + a_1 \left[ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right] \) is the required solution of the given DE.
POWER SERIES METHOD (Frobenious method)

1. Solve by Frobenious method: \( x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0 \), where \( n \) is a non negative real constant or parameter.

>> We assume the series solution of (i) in the form

\[ y = \sum_{r=0}^{\infty} a_r x^{k+r} \quad \text{where} \quad a_0 \neq 0 \]  

(ii)

Hence,

\[ \frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k + r)x^{k+r-1} \]

\[ \frac{d^2 y}{dx^2} = \sum_{r=0}^{\infty} a_r (k + r)(k + r - 1)x^{k+r-2} \]

Substituting these in (i) we get,

\[ x^2 \sum_{r=0}^{\infty} a_r (k + r)(k + r - 1)x^{k+r-2} + x \sum_{r=0}^{\infty} a_r (k + r)x^{k+r} + \sum_{r=0}^{\infty} a_r x^{k+r+1} - n^2 \sum_{r=0}^{\infty} a_r x^{k+r} = 0 \]

i.e., \( \sum_{r=0}^{\infty} a_r (k + r)(k + r - 1)x^{k+r} + \sum_{r=0}^{\infty} a_r (k + r)x^{k+r} + \sum_{r=0}^{\infty} a_r x^{k+r+2} - n^2 \sum_{r=0}^{\infty} a_r x^{k+r} = 0 \)

Grouping the like powers, we get

\[ \sum_{r=0}^{\infty} a_r (k + r)(k + r - 1) + (k + r) - n^2 \sum_{r=0}^{\infty} a_r x^{k+r+2} = 0 \]

(iii)

Now we shall equate the coefficient of various powers of \( x \) to zero

Equating the coefficient of \( x^k \) from the first term and equating it to zero, we get

\[ a_0\left( k^2 - n^2 \right) = 0. \quad \text{Since} \quad a_0 \neq 0, \quad \text{we get} \quad k^2 - n^2 = 0, \quad \therefore k = \pm n \]

Coefficient of \( x^{k+1} \) is got by putting \( r = 1 \) in the first term and equating it to zero, we get
\[ i.e., \ a_i \left( k+1 \right)^2 - n^2 \]  
\[ = 0. \]  
This gives \( a_i = 0, \) since \( (k+1)^2 - n^2 = 0 \) gives, \( k+1 = \pm n \)

which is a contradiction to \( k = \pm n. \)

Let us consider the coefficient of \( x^{k+r} \) from (iii) and equate it to zero.

\[ i.e., \ a_r \left( k+r \right)^2 - n^2 + a_{r-2} = 0. \]

\[ \therefore \ a_r = \frac{-a_{r-2}}{\left( k+r \right)^2 - n^2} \]

(iv)

If \( k = +n, \) (iv) becomes

\[ a_r = \frac{-a_{r-2}}{n + r} - \frac{a_{r-2}}{2 + 2nr} \]

Now putting \( r = 1, 3, 5, \ldots, \) (odd values of \( n \)) we obtain,

\[ a_3 = \frac{-a_1}{6n+9} = 0, \ \therefore \ a_1 = 0 \]

Similarly \( a_5, a_7, \ldots \) are equal to zero.

\[ i.e., \ a_1 = a_5 = a_7 = \ldots = 0 \]

Now, putting \( r = 2, 4, 6 \ldots \) (even values of \( n \)) we get,

\[ a_2 = \frac{-a_0}{4n+4} = \frac{-a_0}{4(n+1)}; \quad a_4 = \frac{-a_2}{8n+16} = \frac{a_0}{32(n+1)(n+2)}; \]

Similarly we can obtain \( a_6, a_8, \ldots \)

We shall substitute the values of \( a_1, a_2, a_3, a_4, \ldots \) in the assumed series solution, we get

\[ y = \sum_{r=0}^{\infty} a_r x^{k+r} = x^k \left( a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \right) \]

Let \( y_1 \) be the solution for \( k = +n \)

\[ \therefore \ y_1 = x^n \left[ a_0 - \frac{a_0}{4(n+1)} x^2 + \frac{a_0}{32(n+1)(n+2)} x^4 - \cdots \right] \]
i.e., \( y_1 = a_0 x^n \left[ 1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^5(n+1)(n+2)} - \cdots \right] \) \( \cdots \) \( \text{ (v)} \)

This is a solution of the Bessel’s equation.

Let \( y_2 \) be the solution corresponding to \( k = -n \). Replacing \( n \) be \( -n \) in (v) we get

\[
y_2 = a_0 x^{-n} \left[ 1 - \frac{x^2}{2^2(-n+1)} + \frac{x^4}{2^5(-n+1)(-n+2)} - \cdots \right] \tag{vi}
\]

The complete or general solution of the Bessel’s differential equation is \( y = c_1 y_1 + c_2 y_2 \), where \( c_1 \), \( c_2 \) are arbitrary constants.

Now we will proceed to find the solution in terms of Bessel’s function by choosing \( a_0 = \frac{1}{2^n(n+1)} \) and let us denote it as \( Y_1 \).

\[
i.e., \quad Y_1 = \frac{x^n}{2^n(n+1)} \left[ 1 - \left( \frac{x}{2} \right)^2 \frac{1}{(n+1)} + \left( \frac{x}{2} \right)^4 \frac{1}{(n+1)(n+2)} - \cdots \right]
\]

We have the result \( \Gamma(n) = (n - 1) \Gamma(n - 1) \) from Gamma function

Hence, \( \Gamma(n + 2) = (n + 1) \Gamma(n + 1) \) and

\[
\Gamma(n + 3) = (n + 2) \Gamma(n + 2) = (n + 2) (n + 1) \Gamma(n + 1)
\]

Using the above results in \( Y_1 \), we get

\[
Y_1 = \left( \frac{x}{2} \right)^n \left[ \frac{1}{(n+1)} - \left( \frac{x}{2} \right)^2 \frac{1}{(n+1)} + \left( \frac{x}{2} \right)^4 \frac{1}{(n+1)(n+2)} - \cdots \right]
\]

which can be further put in the following form

\[
Y_1 = \left( \frac{x}{2} \right)^n \left[ (-1)^0 \frac{x^0}{0!} \left( \frac{x^0}{(n+1)} \right) + (-1)^1 \frac{x^2}{2} \left( \frac{x^2}{(n+2)} \right) + (-1)^2 \frac{x^4}{2} \left( \frac{x^4}{(n+3)} \right) - \cdots \right]
\]
\[
\left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{(n+r+1) \cdot r!} \left(\frac{x}{2}\right)^{2r}
\]

\[
= \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{(n+r+1) \cdot r!}
\]

This function is called the Bessel function of the first kind of order \(n\) and is denoted by \(J_n(x)\).

Thus \(J_n(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{(n+r+1) \cdot r!}\)

Further the particular solution for \(k = -n\) (replacing \(n\) by \(-n\)) be denoted as \(J_{-n}(x)\). Hence the general solution of the Bessel’s equation is given by \(y = AJ_n(x) + BJ_{-n}(x)\), where \(A\) and \(B\) are arbitrary constants.

2. Solve \(2xy'' + 3y' - y = 0\) by Frobenius method.

For the given equation, we seek a series solution in the form

\[
y = \sum_{r=0}^{\infty} a_r x^{m+r}, \quad a_0 \neq 0
\]

From this, we find

\[
\frac{dy}{dx} = \sum_{r=0}^{\infty} (m+r)a_r x^{m+r-1}, \quad \frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} (m+r)(m+r-1)a_r x^{m+r-2}
\]

Substituting for \(y\), \(\frac{dy}{dx}\) and \(\frac{d^2y}{dx^2}\) from the above expressions in the given equation, we get

\[
2 \sum_{r=0}^{\infty} (m+r)(m+r-1)a_r x^{m+r-1} + 3 \sum_{r=0}^{\infty} (m+r)a_r x^{m+r-1} = -\sum_{r=0}^{\infty} a_r x^{m+r} = 0
\]

We observe that in this expression the lowest degree term in \(x\) is \(x^{m-1}\) and this occurs in the first and second series only. Let us rewrite this expression as shown below in which the coefficients of \(x^{m-1}\) and \(x^{m-1+r}\) for \(r \geq 1\) are shown explicitly.

4. Solve by Frobenius method, the equation \(4x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + y = 0\)

soln: For the given equation we seek series solution in the form
\[ y = \sum_{r=0}^{\infty} a_r x^{m+r}, \quad a_0 \neq 0 \quad \text{.........(1)} \]

\[ \frac{dy}{dx} = \sum_{r=0}^{\infty} (m+r)a_r x^{m+r-1}, \quad \frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} (m+r)(m+r-1)a_r x^{m+r-2} \]

Substituting for \( y, \frac{dy}{dx}, \frac{d^2y}{dx^2} \) in the given equation

\[ 4 \sum_{r=0}^{\infty} (m+r)(m+r-1)a_r x^{m+r-1} + 2 \sum_{r=0}^{\infty} (m+r)a_r x^{m+r-1} + \sum_{r=0}^{\infty} a_r x^{m+r} = 0 \]

Rewrite the expression as follows

\[ \left\{ 4m(m-1)a_0 x^{m-1} + 4 \sum_{r=1}^{\infty} (m+r)(m+r-1)a_r x^{m+r-1} \right\} + \]
\[ \left\{ 2ma_0 x^{m-1} + 2 \sum_{r=1}^{\infty} (m+r)a_r x^{m+r-1} \right\} + \sum_{r=1}^{\infty} a_r x^{m+r-1} = 0 \quad \text{.........(2)} \]

Equating the coefficients of \( x^{m-1} \) and \( x^{m+r-1} \) for \( r \geq 1 \) we get

\[ 4m(m-1)a_0 + 2ma_0 = 0 \quad \text{or} \quad 2m^2 - m = 0 \quad \text{.........(3)} \]

and

\[ 4m + r \quad m + r - 1 \quad a_r + 2m + r \quad a_r + a_{r-1} = 0 \]

or

\[ a_r = -\frac{1}{2m+r} \quad a_{r-1} \quad \text{for} \quad r \geq 1 \quad \text{.........(4)} \]

Equation (3) gives \( m_1 = 1/2, \ m_2 = 0 \) \quad \text{.........(5)}

For \( m = m_1 = 1/2 \) equation (4) becomes

\[ a_r = -\frac{1}{2^{1/2+r}} \quad a_{r-1} \quad \text{for} \quad r \geq 1 \quad \text{.........(6)} \]

From this we get

\[ a_1 = -\frac{1}{2 \cdot 3} \quad a_0, \ a_2 = -\frac{1}{4 \cdot 5} \quad a_1, \ a_4 = -\frac{1}{4 \cdot 5 \cdot 23} \quad a_3 = \frac{1}{5!} \quad a_0, \ a_5 = -\frac{1}{7!} \quad a_4, \ \text{and so on} \]

Putting these and \( m = m_1 = 1/2 \) in (1) we have

\[ y = x^m \quad a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots \quad \text{.........} \]

\[ = a_0 x^{1/2} \left\{ 1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \ldots \right\} \quad \text{.........(7)} \]
For \( m = m_2 = 0 \), equation (4) becomes

\[
a_r = -\frac{1}{2r + 2r - 1} a_{r-1} \text{ for } r \geq 1 \ldots (8)
\]

\[
a_1 = -\frac{1}{2} a_0, a_2 = -\frac{1}{4.3} a_1 = \frac{1}{4.3.2} a_0 = \frac{1}{4!} a_0,
\]

\[
a_3 = -\frac{1}{6.5} a_2 = -\frac{1}{6(5.4)!} a_0 = -\frac{1}{6!} a_0, \text{ and so on}
\]

Putting these and \( m = m_2 = 0 \) in (1) we have

\[
y = x^m \left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \ldots \right\}
\]

\[
= a_0 \left\{ 1 - \frac{x}{2} + \frac{x^2}{4!} - \frac{x^3}{6!} + \ldots \right\} \ldots (9)
\]

The solution (7) and (9) are two linearly independent Frobenius type series solution of the given equation, \( a_0 \) is arbitrary constant

A general solution is \( y = A y_1 + B y_2 \) where \( y_1 \) is the solution (7) and \( y_2 \) is the solution (9) and \( A \) & \( B \) are arbitrary constants.

\[
\left\{ 2m(m-1)a_0 x^{m-1} + 2 \sum_{r=1}^{\infty} (m+r)(m+r-1)a_r x^{m+r-1} \right\} + \left\{ 3ma_0 x^{m-1} + 3 \sum_{r=1}^{\infty} (m+r)a_r x^{m+r-1} \right\} - \sum_{r=1}^{\infty} a_{r-1} x^{m+r-1} = 0
\]

Equating the coefficients of \( x^{m-1} \) and \( x^{m-1+r} \) for \( r \geq 1 \) present in the LHS of this expression to zero, we get the following equations:

\[
2m(m-1)a_0 + 3ma_0 = 0,
\]

\[
2(m+r)(m+r-1)a_r + 3(m+r)a_r - a_{r-1} = 0
\]

We note that (iii) is the Indicial equation. Its roots are

\[
m_1 = 0, \ m_2 = -1
\]

which are real and distinct and their difference is not an integer.

Setting \( m = m_1 = 0 \) in (iv) we get

\[
a_r = -\frac{1}{r(2r+1)} a_{r-1} \text{ for } r \geq 1
\]
From this, we get
\[ a_1 = \frac{1}{1.3} a_0, \quad a_2 = \frac{1}{2.5} a_1 = \frac{1}{(1.2)(3.5)} a_0, \]
\[ a_3 = \frac{1}{3.7} a_2 = \frac{1}{(1.2.3)(3.5.7)} a_0 = \frac{1}{3!(3.5.7)} a_0, \]
\[ a_4 = \frac{1}{4.9} a_3 = \frac{1}{(4.9) 3!(3.5.7)} a_0 = \frac{1}{4!(3.5.7.9)} a_0, \]
and so on.

The solutions (vii) and (ix) are two linearly independent solutions of the given equation in series form. In these solutions, \( a_0 \) is an arbitrary constant.

A general solution of the given equation is written down by taking a linear combination of the solutions (vii) and (ix). This means that a general solution of the given equation is
\[ y = Ay_1 + By_2 \]
Where \( y_1 \) is the solution (vii) and \( y_2 \) is the solution (ix) and A and B are arbitrary constants.
UNIT IV

PARTIAL DIFFERENTIAL EQUATIONS (PDE)

Introduction:

Many problems in vibration of strings, heat conduction, electrostatics involve two or more variables. Analysis of these problems leads to partial derivatives and equations involving them. In this unit we first discuss the formation of PDE analogous to that of formation of ODE. Later we discuss some methods of solving PDE.

Definitions:

An equation involving one or more derivatives of a function of two or more variables is called a partial differential equation.

The order of a PDE is the order of the highest derivative and the degree of the PDE is the degree of highest order derivative after clearing the equation of fractional powers.

A PDE is said to be linear if it is of first degree in the dependent variable and its partial derivative.

In each term of the PDE contains either the dependent variable or one of its partial derivatives, the PDE is said to be homogeneous. Otherwise it is said to be a nonhomogeneous PDE.

- Formation of pde by eliminating the arbitrary constants
- Formation of pde by eliminating the arbitrary functions

Solutions to first order first degree pde of the type

\[ P \frac{\partial z}{\partial x} + Q \frac{\partial z}{\partial y} = R \]

Formation of pde by eliminating the arbitrary constants:

(1) Solve: \[ 2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \]

Differentiating (i) partially with respect to x and y,

\[ 2 \frac{\partial z}{\partial x} = \frac{2x}{a^2} \text{ or } \frac{1}{a^2} = \frac{1}{x} \frac{\partial z}{\partial x} = \frac{p}{x} \]
\[
\frac{2\partial z}{\partial y} = \frac{2y}{b^2} \quad \text{or} \quad \frac{1}{b^2} = \frac{1}{y} \frac{\partial z}{\partial x} = \frac{q}{y}
\]

Substituting these values of \(1/a^2\) and \(1/b^2\) in (i), we get

(2) \(2z = x p + y q\)

(3) \(z = (x^2 + a) (y^2 + b)\)

Differentiating the given relation partially

\((x-a)^2 + (y-b)^2 + z^2 = k^2 \ldots\)

Differentiating (i) partially w. r. t. \(x\) and \(y\),

\((x - a) + z \frac{\partial z}{\partial x} = 0, (y - b) + z \frac{\partial z}{\partial y} = 0\)

Substituting for \((x - a)\) and \((y - b)\) from these in (i), we get

\[z^2 \left[ 1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 \right] = k^2 \quad \text{This is the required partial differential equation.}\]

(4) \(z = ax + by + cxy \quad \ldots\) (i)

Differentiating (i) partially w.r.t. \(x\) \(y\), we get

\[\frac{\partial z}{\partial x} = a + cy \quad \ldots\) (ii)

\[\frac{\partial z}{\partial y} = b + cx \quad \ldots\) (iii)

It is not possible to eliminate \(a, b, c\) from relations (i)-(iii).

Partially differentiating (ii),

\[\frac{\partial^2 z}{\partial x \partial y} = c \quad \text{Using this in (ii) and (iii)}\]
\[ a = \frac{\partial z}{\partial x} - y \frac{\partial^2 z}{\partial x \partial y} \]

\[ b = \frac{\partial z}{\partial y} - x \frac{\partial^2 z}{\partial x \partial y} \]

Substituting for \( a, b, c \) in (i), we get

\[ z = x \left[ \frac{\partial z}{\partial x} - y \frac{\partial^2 z}{\partial x \partial y} \right] + y \left[ \frac{\partial z}{\partial y} - x \frac{\partial^2 z}{\partial x \partial y} \right] + xy \frac{\partial^2 z}{\partial x \partial y} \]

\[ z = x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} - xy \frac{\partial^2 z}{\partial x \partial y} \]

(5) \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \]

Differentiating partially w.r.t. \( x \),

\[ \frac{2x}{a^2} + \frac{2z}{c^2} \frac{\partial z}{\partial x} = 0, \text{ or } \frac{x}{a^2} = -\frac{z}{c^2} \frac{\partial z}{\partial x} \]

Differentiating this partially w.r.t. \( x \), we get

\[ \frac{1}{a^2} = -\frac{1}{c^2} \left\{ \left( \frac{\partial z}{\partial x} \right)^2 + z \frac{\partial^2 z}{\partial x^2} \right\} \quad \text{or} \quad \frac{c^2}{a^2} = -\left\{ \left( \frac{\partial z}{\partial x} \right)^2 + z \frac{\partial^2 z}{\partial x^2} \right\} \]

: Differentiating the given equation partially w.r.t. \( y \) twice we get

\[ \frac{z}{y} \frac{\partial z}{\partial y} = \left( \frac{\partial z}{\partial y} \right)^2 + z \frac{\partial^2 z}{\partial y^2} \quad \frac{z}{x} \frac{\partial z}{\partial x} = \left( \frac{\partial z}{\partial x} \right)^2 + z \frac{\partial^2 z}{\partial x^2} \]

Is the required p. d. e..

**Note:**

As another required partial differential equation.

P.D.E. obtained by elimination of arbitrary constants need not be not unique.
Formation of p d e by eliminating the arbitrary functions:

1) \( z = f(x^2 + y^2) \)

Differentiating \( z \) partially w.r.t. \( x \) and \( y \),

\[
\frac{p}{q} = \frac{x}{y} \quad \text{or} \quad y \frac{p}{x} - x q = 0 \quad \text{is the required pde}
\]

(2) \( z = f(x + ct) + g(x - ct) \)

Differentiating \( z \) partially with respect to \( x \) and \( t \),

\[
\frac{\partial z}{\partial x} = f'(x + ct) + g'(x - ct), \quad \frac{\partial^2 z}{\partial x^2} = f''(x + ct) + g''(x - ct)
\]

Thus the pde is

\[
\frac{\partial^2 z}{\partial t^2} + \frac{\partial^2 z}{\partial x^2} = 0
\]

(3) \( x + y + z = f(x^2 + y^2 + z^2) \)

Differentiating partially w.r.t. \( x \) and \( y \)

\[
1 + \frac{\partial z}{\partial x} = f'(x^2 + y^2 + z^2) \left[ 2x + 2z \frac{\partial z}{\partial x} \right]
\]

\[
1 + \frac{\partial z}{\partial y} = f'(x^2 + y^2 + z^2) \left[ 2y + 2z \frac{\partial z}{\partial y} \right]
\]

\[
2f'(x^2 + y^2 + z^2) = \frac{1 + (\partial z / \partial x)}{x + z(\partial z / \partial x)} = \frac{1 + (\partial z / \partial y)}{y + z(\partial z / \partial y)}
\]

\[
(y - z) \frac{\partial z}{\partial x} + (z - x) \frac{\partial z}{\partial y} = x - y \quad \text{is the required pde}
\]
(4) \( z = f(\frac{xy}{z}) \).

Differentiating partially w.r.t. \( x \) and \( y \)

\[
\frac{\partial z}{\partial x} = f'(\frac{xy}{z})\left\{ \frac{y}{z} - \frac{xy}{z^2} \frac{\partial z}{\partial x} \right\}
\]

\[
\frac{\partial z}{\partial y} = f'(\frac{xy}{z})\left\{ \frac{x}{z} - \frac{xy}{z^2} \frac{\partial z}{\partial y} \right\}
\]

\[
f'(\frac{xy}{z}) = \frac{\frac{\partial z}{\partial x}}{(y/z)(1-(x/z)(\frac{\partial z}{\partial x}))} = \frac{\frac{\partial z}{\partial y}}{(x/z)(1-(y/z)(\frac{\partial z}{\partial y}))}
\]

\[
x \frac{\partial z}{\partial x} = y \frac{\partial z}{\partial y}
\]

or \( \text{xp} = \text{yq} \) is the required pde.

(5) \( z = y^2 + 2f(1/x + \log y) \)

\[
\frac{\partial z}{\partial y} = 2y + 2f'(1/x + \log y)\left\{ \frac{1}{y} \right\}
\]

\[
\frac{\partial z}{\partial x} = 2f'(1/x + \log y)\left\{ -\frac{1}{x^2} \right\}
\]

\[
2f'(1/x + \log y) = -x^2 \frac{\partial z}{\partial x} = y\left( \frac{\partial z}{\partial y} - 2y \right)
\]

Hence

\[
x^2 \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2y^2
\]

(6) \( Z = x\Phi(y) + y\psi(x) \)

\[
\frac{\partial z}{\partial x} = \phi(y) + y\psi'(x); \frac{\partial z}{\partial y} = x\phi'(y) + \psi(x)
\]
Substituting \( \phi'(y) \) and \( \psi'(x) \)

\[
xy \frac{\partial^2 z}{\partial x \partial y} = x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} - [x\phi(y) + y\psi(x)]
\]

is the required PDE.

7) Form the partial differential equation by eliminating the arbitrary functions from
\( z = f(y-2x)+g(2y-x) \) (Dec 2011)

**Solution:** By data, \( z = f(y-2x)+g(2y-x) \)

\[
p = \frac{\partial z}{\partial x} = -2f'(y-2x) - g'(2y-x)
\]

\[
q = \frac{\partial z}{\partial y} = f'(y-2x) + 2g'(2y-x)
\]

\[
r = \frac{\partial^2 z}{\partial x^2} = 4f''(y-2x) + g''(2y-x)...............(1)
\]

\[
s = \frac{\partial^2 z}{\partial x \partial y} = -2f''(y-2x) - 2g''(2y-x)...........(2)
\]

\[
t = \frac{\partial^2 z}{\partial y^2} = f''(y-2x) + 4g''(2y-x)............(3)
\]

\((1) \times 2 + (2) \Rightarrow 2r + s = 6f''(y-2x)...........(4)\)

\((2) \times 2 + (3) \Rightarrow 2s + t = -3f''(y-2x)...........(5)\)

*Now dividing (4) by (5) we get*

\[
\frac{2r+s}{2s+t} = -2 \quad \text{or} \quad 2r + 5s + 2t = 0
\]

*Thus* \( 2 \frac{\partial^2 z}{\partial x^2} + 5 \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial^2 z}{\partial y^2} = 0 \) *is the required PDE*
LAGRANGE'S FIRST ORDER FIRST DEGREE PDE: $Pp+Qq=R$

(1) Solve: $yzp + zxq = xy$.

\[
\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}
\]

Subsidiary equations are

From the first two and the last two terms, we get, respectively

\[
\frac{dx}{y} = \frac{dy}{x} \text{ or } xdx - ydy = 0 \quad \text{and} \quad \frac{dy}{z} = \frac{dz}{y} \text{ or } ydy - zdz = 0.
\]

Integrating we get $x^2 - y^2 = a$, $y^2 - z^2 = b$.

Hence, a general solution is

$\Phi (x^2-y^2, y^2-z^2) = 0$

(2) Solve: $y^2p - xyq = x(z-2y)$

\[
\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}
\]

From the first two ratios we get

$x^2 + y^2 = a$ \quad from the last ratios two we get

\[
\frac{dz}{dy} + \frac{z}{y} = 2
\]

from the last ratios two we get

\[
\frac{dz}{dy} + \frac{z}{y} = 2 \text{ ordinary linear differential equation hence}
\]

$yz - y^2 = b$ \quad solution is \quad $\Phi (x^2 + y^2, yz - y^2) = 0$
(3) Solve: \( z(xp - yq) = y^2 - x^2 \)

\[
\frac{dx}{zx} = \frac{dy}{-zy} = \frac{dz}{y^2 - x^2}
\]

\[
\frac{dx}{x} = \frac{dy}{-y}, \quad \text{or} \quad xdy + ydx = 0 \quad \text{or} \quad d(xy) = 0,
\]

on integration, yields \( xy = a \)

\[ xdx + ydy + zdz = 0 \quad x^2 + y^2 + z^2 = b \]

Hence, a general solution of the given equation

\[ \Phi(xy, x^2 + y^2 + z^2) = 0 \]

(4) Solve:

\[
\frac{y-z}{yz} \quad p + \frac{z-x}{zx} \quad q = \frac{x-y}{xy}
\]

\[
\frac{yz}{y-z} \quad dx = \frac{zx}{z-x} \quad dy = \frac{xy}{x-y} \quad dz
\]

\[ x \quad dx + y \quad dy + z \quad dz = 0 \quad \text{(i)} \]

Integrating (i) we get

\[ x^2 + y^2 + z^2 = a \]

\[ yz \quad dx + zx \quad dy + xy \quad dz = 0 \quad \text{(ii)} \]

Dividing (ii) throughout by \( xyz \) and then integrating,

we get \( xyz = b \)

\[ \Phi( x^2 + y^2 + z^2, xyz ) = 0 \]

(5) \((x+2z)p + (4zx - y)q = 2x^2 + y\)

\[
\frac{dx}{x+2z} = \frac{dy}{4zx-y} = \frac{dz}{2x^2+y} \quad \text{(i)}
\]

Using multipliers \( 2x, -1, -1 \) we obtain \( 2x \quad dx - dy - dz = 0 \)
Using multipliers $y, x, -2z$ in (i), we obtain

\[ y \, dx + x \, dy - 2z \, dz = 0 \]

which on integration yields

\[ xy - z^2 = b \quad \text{....(iii)} \]

5) Solve \( z_{xy} = \sin x \sin y \) for which \( z_y = -2 \sin y \) when \( x = 0 \) and \( z = 0 \)

when \( y \) is an odd multiple of \( \frac{\pi}{2} \).

Solution: Here we first find \( z \) by integration and apply the given conditions to determine the arbitrary functions occurring as constants of integration.

The given PDF can be written as

\[ \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \sin x \sin y \]

Integrating w.r.t \( x \) treating \( y \) as constant,

\[ \frac{\partial z}{\partial y} = \sin y \int \sin x \, dx + f(y) = -\sin y \cos x + f(y) \]

Integrating w.r.t \( y \) treating \( x \) as constant

\[ z = -\cos x \int \sin y \, dy + \int f(y) \, dy + g(x) \]

\[ z = -\cos x (-\cos y) + F(y) + g(x), \]

where \( F(y) = \int f(y) \, dy \).

Thus \( z = \cos x \cos y + F(y) + g(x) \)

Also by data, \( \frac{\partial z}{\partial y} = -2 \sin y \) when \( x = 0 \). Use this in (1)

\[ -2 \sin y = (-\sin y) 1 + f(y) (\cos 0 = 1) \]

Hence \( F(y) = \int f(y) \, dy = \int -\sin y \, dy = \cos y \)

With this, (2) becomes \( z = \cos x \cos y + \cos y + g(x) \)

Using the condition that \( z = 0 \) if \( y = (2n + 1) \frac{\pi}{2} \) in (3) we have

\[ 0 = \cos x \cos (2n + 1) \frac{\pi}{2} + \cos x \cos (2n + 1) \frac{\pi}{2} + g(x) \]

But \( \cos (2n + 1) \frac{\pi}{2} = 0 \) and hence \( 0 = 0 + 0 + g(x) \)

Thus the solution of the PDE is given by

\[ z = \cos x \cos y + \cos y \]
Solution of PDE by the method of Separation of Variables

1) Solve by the method of variables $3u_x + 2u_y = 0$, given that $u(x,0) = 4e^{-x}$

Solution: Given $3 \frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial y} = 0$

Assume solution of (1) as

$U = XY$ where $X = X(x); Y = Y(y)$

$3 \frac{\partial u}{\partial x} (xy) + 2 \frac{\partial u}{\partial y} (xy) = 0$

$\Rightarrow 3Y \frac{dX}{dx} + 2 \frac{dY}{dy} = 0 \Rightarrow \frac{dX}{X} = \frac{-2}{Y} \frac{dY}{dy}$

Let $\frac{3}{X} \frac{dX}{dx} = K \Rightarrow \frac{3dX}{X} = kdx$

$\Rightarrow 3\log X = kx + c_1 \Rightarrow \log X = \frac{Kx}{3} + c_1$

$\Rightarrow X = e^{\frac{kx}{3} + c_1}$

Let $\frac{-2}{Y} \frac{dY}{dy} = k \Rightarrow \frac{dY}{Y} = \frac{-Kdy}{2}$

$\Rightarrow \log Y = \frac{-Kdy}{2} + c_2 \Rightarrow Y = e^{\frac{-Ky}{2} + c_2}$

Substituting (2) & (3) in (1)

$U = e^{\frac{k(x+y)}{2} + c_1 + c_2}$

Also $u(x,0) = 4e^{-x}$

i.e., $4e^{-x} = Ae^{\frac{kx}{3}}$ \(\Rightarrow 4e^{-x} = Ae^{\frac{kx}{3}}\)

Comparing we get $A = 4$ & $K = -3$

$U = 4e^{-\left(\frac{x+y}{3} \right)^2}$ is required solution.

2) Solve by the method of variables $4 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$, given that $u(0,y) = 2e^{5y}$ (Dec 2011, June 2012)

Solution: Given $4 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$

Assume solution of (1) as

$u = XY$ where $X = X(x); Y = Y(y)$
\[
4 \frac{\partial}{\partial x} (XY) + \frac{\partial}{\partial y} (XY) = 3XY
\]

\[
\Rightarrow 4Y \frac{dX}{dx} + X \frac{dY}{dy} = 3XY \quad \Rightarrow \frac{4}{X} \frac{dX}{dx} + \frac{1}{Y} \frac{dY}{dy} = 3
\]

Let \( \frac{4}{X} \frac{dX}{dx} = k \), \( 3 - \frac{1}{Y} \frac{dY}{dy} = k \)

Separating variables and integrating we get

\[
\Rightarrow \log X = \frac{kx}{4} + c_1, \quad \log Y = 3 - k \cdot y + c_2
\]

\[
\Rightarrow X = e^{\frac{kx}{4}+c_1} \quad \text{and} \quad Y = e^{3-k \cdot y+c_2}
\]

Hence \( u = XY = e^{c_1+c_2} e^{\frac{kx}{4}+3-k \cdot y} = Ae^{\frac{kx}{4}+3-k \cdot y} \) where \( A = e^{c_1+c_2} \)

Put \( x = 0 \) and \( u = 2e^{5y} \)

The general solution becomes

\[
2e^{5y} = Ae^{3-k \cdot y} \Rightarrow A = 2 \quad \text{and} \quad k = -2
\]

\[\therefore\] Particular solution is

\[
u = 2e^{\frac{x}{2} + 5y}
\]
UNIT V
INTEGRAL CALCULUS

MULTIPLE INTEGRALS

In this topic we discuss a repeated process of integration of a function of two and three variables referred to as

double integrals: \( \iint f(x, y) \, dx \, dy \)

and

triple integrals: \( \iiint f(x, y, z) \, dx \, dy \, dz \).

DOUBLE INTEGRAL

The double integral of a function \( f(x, y) \) over a region \( D \) in \( \mathbb{R}^2 \) is denoted by \( \iint_D f(x, y) \, dx \, dy \).

Let \( f(x, y) \) be a continuous function in \( \mathbb{R}^2 \) defined on a closed rectangle

\[ R = \{(x, y) | a \leq x \leq b \text{ and } c \leq y \leq d \} \]

For any fixed \( x \in [a, b] \) consider the integral \( \int_c^d f(x, y) \, dy \).

The value of this integral depends on \( x \) and we get a new function of \( x \). This can be integrated depends on \( x \) and, we get \( \int_a^b \left[ \int_c^d f(x, y) \, dy \right] \, dx \). This is called an “iterated integral”.

Similarly, we can define another

\[ \int_c^d \left[ \int_a^b f(x, y) \, dx \right] \, dy \]

For continuous function \( f(x, y) \), we have

\[ \iint_D f(x, y) \, dx \, dy = \int_a^b \left[ \int_c^d f(x, y) \, dy \right] \, dx = \int_c^d \left[ \int_a^b f(x, y) \, dx \right] \, dy \]

If \( f(x, y) \) is continuous on a bounded region \( S \) and \( S \) is given by

\[ S = \{(x, y) | a \leq x \leq b \text{ and } \phi_1(x) \leq y \leq \phi_2(x) \} \], where \( \phi_1 \) and \( \phi_2 \) are two continuous functions on \([a, b]\) then
The iterated integral in the R.H.S. is also written in the form

\[
\int_{a}^{b} \left[ \int_{\phi_1(x)}^{\phi_2(x)} f(x, y) \, dy \right] \, dx
\]

Similarly, if \( S = \{(x, y)/c \leq y \leq d \) and \( \phi_1(y) \leq x \leq \phi_2(y)\) \)

then \[
\int_{a}^{b} \int_{\phi_1(x)}^{\phi_2(x)} f(x, y) \, dy \, dx
\]

If \( S \) cannot be written in neither of the above two forms we divide \( S \) into finite number of sub-regions such that each of the subregions can be represented in one of the above forms and we get the double integral over \( S \) by adding the integrals over these subregions.

**PROBLEMS:**

1. Evaluate: \( I = \int_{0}^{1} \int_{0}^{2} xy^2 \, dy \, dx \).

   Solution
   
   \[
   I = \int_{0}^{1} \left[ \int_{0}^{2} xy^2 \, dy \right] \, dx
   \]
   
   \[
   = \int_{0}^{1} \left[ \frac{xy^3}{3} \right]_{0}^{2} \, dx
   \]
   
   (Integrating w.r.t. \( y \) keeping \( x \) constant)
   
   \[
   = \frac{1}{3} \int_{0}^{1} 8x \, dx
   \]
   
   \[
   = \frac{1}{3} \left[ 8x^2 \right]_{0}^{1} = \frac{4}{3}
   \]

2. Evaluate: \( \int_{0}^{1} \int_{0}^{2} xy \, dy \, dx \).

   Solution. Let \( I \) be the given integral

   Then, \[
   I = \int_{0}^{1} x \left\{ \int_{0}^{2} y \, dy \right\} \, dx
   \]
   
   \[
   = \int_{0}^{1} x \left[ \frac{y^2}{2} \right]_{0}^{2} \, dx
   = \frac{3}{2} \int_{0}^{1} x \, dx = \frac{3}{4}
   \]
5. Evaluate: \[ \int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} \left( x^2 + y^2 + z^2 \right) dz \, dy \, dx. \]

Solution \[
I = \int_{x=-c}^{c} \int_{y=-b}^{b} \int_{z=-a}^{a} \left( x^2 + y^2 + z^2 \right) dz \, dy \, dx
\]

Integrating w.r.t. \( z \), \( x \) and \( y \) – constant.

\[
= \int_{x=-c}^{c} \int_{y=-b}^{b} \left[ x^2 z + y^2 z + \frac{z^3}{3} \right]_{z=-a}^{a} \, dy \, dx
\]

\[
= \int_{x=-c}^{c} \int_{y=-b}^{b} \left[ x^2 (a + a) + y^2 (a + a) + \left( \frac{a^3}{3} + \frac{a^3}{3} \right) \right] \, dy \, dx
\]

\[
= \int_{x=-c}^{c} \int_{y=-b}^{b} \left( 2ax^2 + 2ay^2 + \frac{2a^3}{3} \right) \, dy \, dx
\]

Integrating w.r.t. \( y \), \( x \) – constant.

\[
= \int_{x=-c}^{c} \left[ 2ax^2 y + \frac{2ay^3}{3} + \frac{2a^3}{3} y \right]_{y=-b}^{b} \, dx
\]

\[
= \int_{x=-c}^{c} \left[ 2ax^2 (b + b) + \frac{2a}{3} (b^3 + b^3) + \frac{2a^3}{3} (b + b) \right] \, dx
\]

\[
= \int_{x=-c}^{c} \left( 4ax^2 b + \frac{4a b^3}{3} + \frac{4a^3 b}{3} \right) \, dx
\]

\[
= \left[ 4ab \left( \frac{x^3}{3} \right) + \frac{4a b^3}{3} (x) + \frac{4a^3 b}{3} (x) \right]_{c}^{c}
\]

\[
= 4ab \left( \frac{2c^3}{3} \right) + \frac{4a b^3}{3} (2c) + \frac{4a^3 b}{3} (2c)
\]

\[
= \frac{8abc^3}{3} + \frac{8ab^3 c}{3} + \frac{8a^3 bc}{3}
\]

\[
I = \frac{8abc}{3} \left( a^2 + b^2 + c^2 \right).
\]
Evaluation of a Double Integral by Changing the Order of Integration

In the evaluation of the double integrals sometimes we may have to change the order of integration so that evaluation is more convenient. If the limits of integration are variables then change in the order of integration changes the limits of integration. In such cases a rough idea of the region of integration is necessary.

Evaluation of a Double Integral by Change of Variables

Sometimes the double integral can be evaluated easily by changing the variables.

Suppose $x$ and $y$ are functions of two variables $u$ and $v$.

\[ i.e., \quad x = x(u, v) \text{ and } y = y(u, v) \text{ and the Jacobian} \]

\[ J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0 \]

Then the region $A$ changes into the region $R$ under the transformations

\[ x = x(u, v) \text{ and } y = y(u, v) \]

Then

\[ \int \int_A f(x, y) \, dx \, dy = \int \int_R f(u, v) \, J \, du \, dv \]

If

\[ x = r \cos \theta, \quad y = r \sin \theta \]

\[ J = \begin{vmatrix} \frac{\partial (x, u)}{\partial (r, \theta)} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r \]

\[ \therefore \quad \int \int_A f(x, y) \, dx \, dy = \int \int_R F(r, \theta) \, r \, dr \, d\theta. \quad \ldots(1) \]

Applications to Area and Volume

1. \[ \int \int_R \, dx \, dy = \text{Area of the region } R \text{ in the Cartesian form.} \]

2. \[ \int \int_R r \cdot dr \, d\theta = \text{Area of the region } R \text{ in the polar form.} \]

3. \[ \int \int \int_V \, dx \, dy \, dz = \text{Volume of a solid.} \]

4. Volume of a solid (in polars) obtained by the revolution of a curve enclosing an area $A$ about the initial line is given by

\[ V = \int_A 2\pi r^2 \sin \theta \cdot dr \, d\theta. \]
5. If \( z = f(x, y) \) be the equation of a surface \( S \) then the surface area is given by

\[
\iint_A \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \, dx \, dy
\]

**Type 1. Evaluation over a given region**

1. Evaluate \( \iint_R xy \, dx \, dy \) where \( R \) is the triangular region bounded by the axes of coordinates and the line \( \frac{x}{a} + \frac{y}{b} = 1 \).

**Solution.** \( R \) is the region bounded by \( x = 0, y = 0 \) being the coordinates axes and \( \frac{x}{a} + \frac{y}{b} = 1 \) being the straight line through \((0, a)\) and \(0, b \left(1 - \frac{x}{a}\right)\) when \( x \) is held fixed and \( y \) varies from 0 to \( b \left(1 - \frac{x}{a}\right) \)

\[
\therefore \quad \frac{x}{a} + \frac{y}{b} = 1
\]

\[
\Rightarrow \quad \frac{y}{b} = 1 - \frac{x}{a}
\]

\[
\Rightarrow \quad y = b \left(1 - \frac{x}{a}\right)
\]

\[
\therefore \quad \int \int_R xy \, dx \, dy = \int_0^a \left\{ \int_0^{b \left(1 - \frac{x}{a}\right)} xy \, dy \right\} \, dx
\]

\[
= \int_0^a \left[ \frac{y^2}{2} \right]_0^{b \left(1 - \frac{x}{a}\right)} \, dx
\]

\[
= \int_0^a \left[ \frac{x \cdot b^2}{2} \left(1 - \frac{x}{a}\right)^2 \right] \, dx
\]

\[
= \frac{b^2}{2} \int_0^a \left( x - 2\frac{x^2}{a} + \frac{x^3}{a^2} \right) \, dx
\]

\[
= \frac{b^2}{2} \left[ x^2 - 2\frac{x^3}{3a} + \frac{x^4}{4a^2} \right]_0^a
\]

\[
= \frac{b^2}{2} \left[ a^2 - 2\frac{a^3}{3a} + \frac{a^4}{4a^2} \right]
\]
\[ b^2 \left[ \frac{a^2}{2} - \frac{2}{3} a^2 + \frac{1}{4} a^2 \right] \]
\[ = \frac{a^2 b^2}{24} \]

2. Evaluate \( \int \int xy \, dx \, dy \) over the area in the first quadrant bounded by the circle \( x^2 + y^2 = a^2 \).

Solution
\[
\int \int xy \, dx \, dy = \int_{x=0}^{a} \left[ \int_{y=0}^{\sqrt{a^2-x^2}} xy \, dy \right] \, dx
\]
\[ = \int_{x=0}^{a} \left[ \frac{y^2}{2} \right]_{0}^{\sqrt{a^2-x^2}} \, dx \]
\[ = \int_{x=0}^{a} \left( \frac{a^2-x^2}{2} \right) \, dx \]
\[ = \frac{1}{2} \int_{0}^{a} (a^2-x^2) \, dx \]
\[ = \frac{1}{2} \left[ \frac{a^2 x^2}{2} - \frac{x^4}{4} \right]_{0}^{a} \]
\[ = \frac{1}{2} \left[ \frac{a^4}{2} - \frac{a^4}{4} \right] = \frac{a^4}{8} \]

Type 2. Evaluation of a double integral by changing the order of integration

1. Change the order of integration and hence evaluate \( \int_{0}^{a} \int_{0}^{2 \sqrt{ax}} x^2 \, dy \, dx \).

Solution
\[ y = 2 \sqrt{ax} \]
\[ \Rightarrow \]
\[ y^2 = 4ax \]
when \( x = a \) on \( y^2 = 4ax \), \( y^2 = 4a^2 \)
\[ \Rightarrow \]
\[ y = \pm 2a \]

So, on \( y = 2 \sqrt{ax} \), \( y = 2a \) when \( x = a \)

The integral is over the shaded region.
\[ \int_0^a \int_0^{2\sqrt{ax}} x^2 \, dy \, dx = \int_0^{a^2/4a} \int_y^{a^2/4a} x^2 \, dx \, dy \]

(By changing the order)

\[ = \int_0^{2a} \left[ \frac{x^3}{3} \right]_{y^2/4a}^{a^2/4a} \, dy \]

\[ = \int_0^{2a} \left( \frac{a^3}{3} - \frac{y^6}{192a^3} \right) \, dy \]

\[ = \left[ \frac{a^3}{3} y - \frac{y^7}{192a^3 \times 7} \right]_0^{2a} \]

\[ = \frac{2a^4}{3} - \frac{2^7 a^4}{192 \times 7} \]

\[ = a^4 \left( \frac{2}{3} - \frac{2}{21} \right) = \frac{4}{7} a^4. \]
2. Change the order of integration and hence evaluate \[ \int_0^1 \int_0^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} \, dy \, dx. \]

Solution \[ y = \sqrt{2-x^2} \]
\[ \Rightarrow y^2 = 2-x^2 \]
\[ \Rightarrow x^2 + y^2 = 2 \]

This circle and \( y = x \) meet if \( x^2 + x^2 = 2 \)
\[ \Rightarrow 2x^2 = 2 \Rightarrow x = 1 \]

So, \((1, 1)\) is the meeting point.

Now
\[ I = \int_0^1 \int_0^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} \, dy \, dx \]
\[ = \int_0^{\sqrt{2}} \int_{y=0}^{x=0} \frac{x}{\sqrt{x^2+y^2}} \, dx \, dy \]

where \( \phi(y) = \begin{cases} y & \text{for } 0 \leq y \leq 1 \\ \sqrt{2-y^2} & \text{for } 1 \leq y \leq \sqrt{2} \end{cases} \)

(Note that \( x = \phi(y) \) is the R.H.S. boundary of the shaded region)

So, the required integral is
\[ I = \int_0^{\sqrt{2}} \int_{y=0}^{x=0} \frac{x}{\sqrt{x^2+y^2}} \, dx \, dy + \int_0^{\sqrt{2}} \int_{x=0}^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}} \, dx \, dy \]
\[ = \int_0^1 \left[ x^2 + y^2 \right]_0^y dy + \int_1^\sqrt{2} \left[ \sqrt{x^2+y^2} \right]_0^{\sqrt{2-y^2}} dy \]
\[ = \int_0^1 (\sqrt{2} - y) \, dy + \int_1^\sqrt{2} (\sqrt{2} - y) \, dy \]
\[
= \left[ \left( \frac{\sqrt{2} - 1}{2} \right) \frac{y^2}{2} \right]_0^1 + \left[ \sqrt{2}y - \frac{y^2}{2} \right]_{y=0}^{y=1} \\
= \frac{\sqrt{2} - 1}{2} + \sqrt{2} (\sqrt{2} - 1) - \left( \frac{2}{2} - \frac{1}{2} \right) \\
= 1 - \frac{1}{\sqrt{2}}.
\]

**Type 3. Evaluation by changing into polars**

1. Evaluate \( \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy \) by changing to polar coordinates.

**Solution.** In polars we have \( x = r \cos \theta, y = r \sin \theta \)

\[x^2 + y^2 = r^2 \quad \text{and} \quad dx \, dy = r \, dr \, d\theta \]

Since \( x, y \) varies from 0 to \( \infty \)

\( r \) also varies from 0 to \( \infty \)

In the first quadrant \( \theta \)

varies from 0 to \( \pi/2 \)

Thus

\[I = \int_0^{\pi/2} \int_0^\infty e^{-r^2} \, r \, dr \, d\theta \]

Put

\[r^2 = t \quad \therefore \quad r \, dr = \frac{dt}{2} \]

\( t \) also varies from 0 to \( \infty \)

\[I = \int_0^{\pi/2} \int_0^\infty e^{-t} \frac{dt}{2} \, d\theta \]

\[= \frac{1}{2} \int_0^{\pi/2} \left[ -e^{-t} \right]_0^\infty \, d\theta \]

\[= \frac{-1}{2} \int_0^{\pi/2} (0 - 1) \, d\theta \]

\[= \frac{-1}{2} \int_0^{\pi/2} 1 \, d\theta \]

\[= \frac{-1}{2} \left[ \theta \right]_0^{\pi/2} = \frac{-1}{2} \cdot \frac{\pi}{2} = \frac{-\pi}{4} \]
2. Evaluate \( \int_{0}^{a} \int_{0}^{y} \sqrt{x^2 + y^2} \, dx \, dy \) by changing into polars.

**Solution**

\[
I = \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} r^2 \sin \theta \, r \, dr \, d\theta
\]

\[
x = \sqrt{a^2 - y^2} \quad \text{or} \quad x^2 + y^2 = a^2 \quad \text{is a circle with centre origin and radius} \ a. \ \text{Since}, \ y \ \text{varies from} \ 0 \ \text{to} \ a \\
\text{the region of integration is the first quadrant of the circle.}
\]

In polars, we have \( x = r \cos \theta, \ y = r \sin \theta \)

\[
\therefore \quad x^2 + y^2 = r^2
\]

\[i.e., \quad r^2 = a^2\]

\[\Rightarrow \quad r = a\]

Also \( x = 0, \ y = 0 \) will give \( r = 0 \) and hence we can say that \( r \) varies from 0 to \( a \). In the first quadrant \( \theta \) varies from 0 to \( \pi/2 \), we know that \( dx \, dy = r \, dr \, d\theta \)

\[
\therefore \quad I = \int_{r=0}^{a} \int_{\theta=0}^{\pi/2} r^3 \sin \theta \, dr \, d\theta
\]

\[
= \int_{r=0}^{a} r^3 (\cos \theta)_{0}^{\pi/2} \, dr
\]

\[
= \int_{r=0}^{a} r^3 (0 - 1) \, dr = \left[ \frac{r^4}{4} \right]_{0}^{a} = \frac{a^4}{4}
\]

\[
I = \frac{a^4}{4}
\]
BETA AND GAMMA FUNCTIONS

In this topic we define two special functions of improper integrals known as Beta function and Gamma function. These functions play important role in applied mathematics.

Definitions

1. The Beta function denoted by \( \beta (m, n) \) or \( \beta (m, n) \) is defined by
   \[
   \beta (m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx, \quad (m, n > 0)
   \]
   \( \ldots (1) \)

2. The Gamma function denoted by \( \Gamma (n) \) is defined by
   \[
   \Gamma (n) = \int_0^\infty x^{n-1} e^{-x} \, dx
   \]
   \( \ldots (2) \)

Properties of Beta and Gamma Functions

1. \( \beta (m, n) = \beta (n, m) \)

2. \[
\beta (m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} \, dx
\]
   \( \ldots (3) \)

3. \[
\beta (m, n) = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta \, d\theta
\]
   \( \ldots (4) \)
   \[
   = 2 \int_0^{\pi/2} \sin^{2n-1} \theta \cos^{2m-1} \theta \, d\theta
   \]
4. \[ \beta \left( \frac{p+1}{2}, \frac{q+1}{2} \right) = 2 \int_0^{\pi/2} \sin^p \theta \cos^q \theta \, d\theta \]

\[ = 2 \int_0^{\pi/2} \sin^q \theta \cos^p \theta \, d\theta \quad \ldots(5) \]

5. \[ \Gamma (n + 1) = n \Gamma (n) \quad \ldots(6) \]

6. \[ \Gamma (n + 1) = n!, \text{ if } n \text{ is a positive real number.} \]

**Proof 1.** We have

\[ \beta (m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx \]

\[ = \int_0^1 (1-x)^{m-1} \left[ 1-(1-x) \right]^{n-1} \, dx \]

Since \[ \int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx \]

\[ = \int_0^1 (1-x)^{m-1} (1-1+x)^{n-1} \, dx \]

\[ = \int_0^1 x^{n-1} (1-x)^{m-1} \, dx \]

\[ = \beta (n, m) \]

Thus, \[ \beta (m, n) = \beta (n, m) \]

Hence (1) is proved.

(2) By definition of Beta function,

\[ \beta (m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx \]

Substituting \[ x = \frac{1}{1+t} \text{ then } dx = -\frac{1}{(1+t)^2} \, dt \text{ when } x = 0, t = \infty \text{ and when } x = 1, t = 0. \]

Therefore,

\[ \beta (m, n) = \int_\infty^0 \left( \frac{1}{1+t} \right)^{m-1} \left[ 1-\frac{1}{1+t} \right]^{n-1} \left( -\frac{1}{(1+t)^2} \right) \, dt \]
\[
\beta(m, n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} \, dt = \int_0^\infty \frac{x^{n-1}}{(1+x)^{n+m}} \, dx
\]

Similarly, \( \beta(n, m) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx \)

Since, \( \beta(m, n) = \beta(n, m) \), we get

\[
\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{n+m}} \, dx
\]

(3) By definition of Beta functions

\[
\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx
\]

Substitute \( x = \sin^2 \theta \) then \( dx = 2 \sin \theta \cos \theta \, d\theta \)

Also when \( x = 0, \theta = 0 \)

when \( x = 1, \theta = \frac{\pi}{2} \)

\[
\therefore \beta(m, n) = \int_0^{\pi/2} \left( \sin^2 \theta \right)^{m-1} \left( 1 - \sin^2 \theta \right)^{n-1} 
2 \sin \theta \cos \theta \, d\theta
\]

\[
= 2 \int_0^{\pi/2} \sin^{2n-2} \theta \cos^{2n-2} \theta \sin \theta \cos \theta \, d\theta
\]

\[
= 2 \int_0^{\pi/2} \sin^{2n-1} \theta \cos^{2n-1} \theta \, d\theta
\]
Since, \( \beta (m, n) = \frac{\pi}{2} \int_{0}^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta \, d\theta \), we have

\[
\beta (m, n) = 2 \int_{0}^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta \, d\theta
\]

So that \( m = \frac{p + 1}{2}, \quad n = \frac{q + 1}{2} \) in the above result, we have

\[
\beta \left( \frac{p + 1}{2}, \frac{q + 1}{2} \right) = 2 \int_{0}^{\pi/2} \sin^{p} \theta \cos^{q} \theta \, d\theta
\]

(4) Substituting \( 2m - 1 = p \) and \( 2n - 1 = q \)

So that \( m = \frac{p + 1}{2}, \quad n = \frac{q + 1}{2} \) in the above result, we have

\[
\beta \left( \frac{p + 1}{2}, \frac{q + 1}{2} \right) = 2 \int_{0}^{\pi/2} \sin^{p} \theta \cos^{q} \theta \, d\theta
\]

(1) Substituting \( q = 0 \) in the above result, we get

\[
\beta \left( \frac{p + 1}{2}, \frac{1}{2} \right) = 2 \int_{0}^{\pi/2} \sin^{p} \theta \, d\theta = 2 \int_{0}^{\pi/2} \cos^{p} \theta \, d\theta.
\]

(2) Substituting \( p = 0 \) and \( q = 0 \) in the above result

\[
\beta \left( \frac{1}{2}, \frac{1}{2} \right) = 2 \int_{0}^{\pi/2} \, d\theta = \pi
\]

(5) Replacing \( n \) by \( (n + 1) \) in the definition of gamma function.

\[
\Gamma (n) = \int_{0}^{\infty} x^{n-1} \cdot e^{-x} \, dx
\]

where \( n = (n + 1) \)

\[
\Gamma (n + 1) = \int_{0}^{\infty} x^{n} \cdot e^{-x} \, dx
\]

On integrating by parts, we get

\[
\Gamma (n + 1) = \left[ x^{n} \cdot (-e^{-x}) \right]_{0}^{\infty} - \int_{0}^{\infty} (-e^{-x}) \cdot n x^{n-1} \, dx
\]

\[
= 0 + n \int_{0}^{\infty} e^{-x} x^{n-1} \, dx = n \Gamma (n).
\]
Thus,
\[ \Gamma(n + 1) = n \Gamma(n), \text{ for } n > 0 \]

This is called the recurrence formula, for the gamma function.

(6) If \( n \) is a positive integer then by repeated application of the above formula, we get

\[
\begin{align*}
\Gamma(n + 1) &= n \Gamma(n) \\
&= n \Gamma(n - 1 + 1) \\
&= n (n - 1) \Gamma(n - 1) \text{ (using above result)} \\
&= n (n - 1) (n - 2) \Gamma(n - 2) \\
& \quad \vdots \\
&= n (n - 1) (n - 2) \cdots (1) \Gamma(1) \\
&= n! \Gamma(1)
\end{align*}
\]

But
\[
\Gamma(1) = \int_0^\infty x^0 e^{-x} \, dx
\]

\[
= -[e^{-x}]_0^\infty = -(0 - 1) = 1
\]

Hence
\[ \Gamma(n + 1) = n!, \text{ if } n \text{ is a positive integer.} \]

For example
\[
\begin{align*}
\Gamma(2) &= 1! = 1, \quad \Gamma(3) = 2! = 2, \quad \Gamma(4) = 3! = 6 \\
\end{align*}
\]

If \( n \) is a positive fraction then using the recurrence formula \( \Gamma(n + 1) = n \Gamma(n) \) can be evaluated as follows.

(1) \[
\Gamma\left(\frac{3}{2}\right) = \Gamma\left(1 + \frac{1}{2}\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right)
\]

(2) \[
\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2} + 1\right) = \frac{3}{2} \Gamma\left(\frac{3}{2}\right)
\]

(3) \[
\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2} + 1\right) = \frac{5}{2} \Gamma\left(\frac{5}{2}\right)
\]

\[
= \frac{5 \cdot 3 \cdot 1}{2 \cdot 2 \cdot \frac{1}{2}} \Gamma\left(\frac{1}{2}\right) = \frac{15}{8} \Gamma\left(\frac{1}{2}\right).
\]
Relationship between Beta and Gamma Functions

The Beta and Gamma functions are related by

\[ \beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \]  

... (7)

Proof. We have

\[ \Gamma(n) = \int_0^\infty x^{n-1} \cdot e^{-x} \, dx \]

Substituting \( x = t^2 \), \( dx = 2t \, dt \), we get

\[ \Gamma(n) = \int_0^\infty (t^2)^{n-1} \cdot e^{-t^2} \cdot 2t \, dt \]

\[ = 2 \int_0^\infty t^{2n-1} \cdot e^{-t^2} \, dt \]

\[ \Gamma(n) = 2 \int_0^\infty x^{2n-1} \cdot e^{-x} \, dx \]  

... (i)

Replacing \( n \) by \( m \), and \( x \) by \( y \), we have

\[ \Gamma(m) = 2 \int_0^\infty y^{2m-1} \cdot e^{-y^2} \, dy \]  

... (ii)

Hence

\[ \Gamma(m) \cdot \Gamma(n) = \left\{ 2 \int_0^\infty x^{2n-1} \cdot e^{-x^2} \, dx \right\} \cdot \left\{ 2 \int_0^\infty y^{2m-1} \cdot e^{-y^2} \, dy \right\} \]

\[ = 4 \int_0^\infty \int_0^\infty x^{2n-1} \cdot e^{-x^2} \cdot y^{2m-1} \cdot e^{-y^2} \, dx \, dy \]
\[
\int_{0}^{\infty} \int_{0}^{\infty} e^{-x^2 - y^2} x^{m-1} y^{n-1} \, dx \, dy = 4 \int_{0}^{\pi} \int_{0}^{\frac{\pi}{2}} e^{-r^2} (r \cos \theta)^{2m-1} (r \sin \theta)^{2n-1} \, r \, d\theta \, dr  
\]

We shall transform the double integral into polar coordinates.
Substitute \( x = r \cos \theta, y = r \sin \theta \) then we have \( dx \, dy = r \, dr \, d\theta \)
As \( x \) and \( y \) varies from 0 to \( \infty \), the region of integration entire first quadrant. Hence, \( \theta \) varies from \( 0 \) to \( \frac{\pi}{2} \) and \( r \) varies from 0 to \( \infty \) and also \( x^2 + y^2 = r^2 \)
Hence \((iii)\) becomes,
\[
\Gamma(m) \Gamma(n) = 4 \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r^2} (r \cos \theta)^{2m-1} (r \sin \theta)^{2n-1} \, r \, d\theta \, dr
\]
\[
= 4 \int_{r=0}^{\infty} r^{2(m+n)-1} e^{-r^2} \, dr \times \int_{0}^{\pi/2} \sin^{2n-1} \theta \cos^{2m-1} \theta \, d\theta
\]

Substituting \( r^2 = t \), in the first integral. We get,
\[
\int_{r=0}^{\infty} r^{2(m+n)-1} e^{-r^2} \, dr = \frac{1}{2} \int_{0}^{\infty} t^{m+n-1} e^{-t} \, dt
\]
\[
= \frac{1}{2} \Gamma(m+n)
\]

and from \((iv)\),
\[
\int_{0}^{\pi/2} \sin^{2n-1} \theta \cos^{2m-1} \theta \, d\theta = \frac{1}{2} \beta(m,n)
\]

Therefore \((iv)\) reduces to \( \Gamma(m) \Gamma(n) = \Gamma(m+n) \beta(m,n) \)

Thus,
\[
\beta(m,n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \quad \text{Hence proved.}
\]

Corollary. To show that \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \)

Putting \( m = n = \frac{1}{2} \) in this result, we get
\[
\beta\left[\frac{1}{2}, \frac{1}{2}\right] = \frac{\Gamma\left[\frac{1}{2}\right] \cdot \Gamma\left[\frac{1}{2}\right]}{\Gamma[1]}
\]

But \( \Gamma(1) = 1 \)
\[
\therefore \quad \beta\left[\frac{1}{2}, \frac{1}{2}\right] = \left(\Gamma\left(\frac{1}{2}\right)\right)^2
\]

\( \ldots(8) \)
Now consider $\beta(m, n) = \frac{\pi}{2} \int_0^\pi \sin^{2n-1}\theta \cos^{2m-1}\theta \, d\theta$

Now we have from (8), L.H.S.

$$\beta\left[\frac{1}{2}, \frac{1}{2}\right] = 2 \int_0^{\frac{\pi}{2}} \sin^0\theta \cos^0\theta \, d\theta = 2 \left[\theta\right]_0^{\frac{\pi}{2}} = \pi$$

$$\pi = \Gamma\left(\frac{1}{2}\right)^2 \therefore \Gamma\left[\frac{1}{2}\right] = \sqrt{\pi}.$$

Prove that $\int_0^\infty a^{-bx^2} \, dx = \frac{\sqrt{\pi}}{2\sqrt{b \log a}}$ where $a$ and $b$ are positive constants.

1. Now, $\int_0^\infty a^{-bx^2} \, dx = \int_0^\infty \left[e^{\log a}\right]^{-bx^2} \, dx$ since $a = e^{\log a}$

$$= \int_0^\infty e^{-\left(b \log a\right)x^2} \, dx$$

Substitute $(b \log a) x^2 = t, \, dx = \frac{dt}{(b \log a) \cdot 2x}$

So that, $x = \frac{\sqrt{t}}{\sqrt{b \log a}}$

$\therefore \quad dx = \frac{dt}{2\sqrt{t} \sqrt{b \log a}}$

$$\int_0^\infty e^{-bx^2} \, dx = \int_0^\infty e^{-t} \cdot \frac{dt}{2\sqrt{t} \sqrt{b \log a}}$$
\[
\frac{1}{2\sqrt{b \log a}} \int_{0}^{\infty} t^{\frac{1}{2}} e^{-t} \, dt
\]
\[
= \frac{1}{2\sqrt{b \log a}} \int_{0}^{\infty} \frac{1}{2} t^{\frac{1}{2}-1} e^{-t} \, dt
\]
\[
= \frac{1}{2\sqrt{b \log a}} \Gamma \left( \frac{1}{2} \right)
\]
\[
= \frac{\sqrt{\pi}}{2\sqrt{b \log a}}.
\]

Prove that \( \int_{0}^{\infty} x^{m} e^{-ax} \, dx = \frac{1}{na} \Gamma \left( \frac{m+1}{n} \right) \), where \( m \) and \( n \) are positive constants.

2.

Substitute \( ax^n = t \) so that \( x = \left( \frac{t}{a} \right)^{\frac{1}{n}} \)

Then
\[
\frac{dx}{dt} = \frac{1}{na} \frac{1}{t^{\frac{1}{n}-1}}
\]

Therefore,
\[
\int_{0}^{\infty} x^{m} e^{-ax^n} \, dx = \int_{0}^{\infty} \left[ \left( \frac{t}{a} \right)^{\frac{1}{n}} \right]^{mn} \frac{t^{\frac{1}{n}-1}}{na} \cdot \frac{1}{t^{\frac{1}{n}-1}} e^{-t} \, dt
\]
\[
= \frac{1}{na} \int_{0}^{\infty} t^{\frac{m+1}{n}} e^{-t} \, dt
\]
\[
= \frac{1}{na} \Gamma \left( \frac{m+1}{n} \right).
\]
UNIT VI

VECTOR CALCULUS

Vector Line Integral

If \( \vec{F} \) is a force acting on a particle at a point \( P \) whose positive vector is \( r \) on a curve \( C \) then \( \int_C \vec{F} \cdot d\vec{r} \) represents physically the total work done in moving the particle along \( C \).

Thus, total work done is \( \int_C \vec{F} \cdot d\vec{r} = 0 \)

PROBLEMS:

1. If \( \vec{F} = (5xy - 6x^2) \hat{i} + (2y - 4x) \hat{j} \). Evaluate \( \int_C \vec{F} \cdot d\vec{r} \) along the curve \( y = x^3 \) in the \( x-y \) plane from \((1, 1)\) to \((2, 8)\).

Solution. We have \( \vec{F} = (5xy - 6x^2) \hat{i} + (2y - 4x) \hat{j} \) and \( \vec{r} = xi + yj \) will give

\[
\frac{dr}{dt} = dx \hat{i} + dy \hat{j}
\]

\[ \therefore \quad \vec{F} \cdot d\vec{r} = (5xy - 6x^2) \, dx + (2y - 4x) \, dy \]

Since \( y = x^3 \) we have \( dy = 3x^2 \, dx \) and varies from \( 1 \) to \( 2 \)

\[
\int_C \vec{F} \cdot d\vec{r} = (5x \cdot x^3 - 6x^2) \, dx + (2 \cdot x^3 - 4x) \cdot 3x^2 \, dx
\]

\[
= \int_1^2 (5x^4 - 6x^2 + 6x^5 - 12x^3) \, dx
\]

\[
= \left[ x^5 - 2x^3 + x^6 - 3x^4 \right]_1^2 = 35
\]

2. If \( \vec{F} = (3x^2 + 6y) \hat{i} - 14yz \hat{j} + 20xz^2 \hat{k} \). Evaluate \( \int_C \vec{F} \cdot d\vec{r} \) from \((0, 0, 0)\) to \((1, 1, 1)\) along the path \( x = t, \ y = t^2, \ z = t^3 \).
Solution

\[ \vec{F} = (3x^2 + 6y) \vec{i} - 14yz \vec{j} + 20xz^2 \vec{k} \]

\[ \vec{\alpha} = dx \vec{i} + dy \vec{j} + dz \vec{k} \]

\therefore

\[ \vec{F} \cdot \vec{dr} = (3x^2 + 6y) \, dx - 14yz \, dy + 20xz^2 \, dz \]

Since

\[ x = t, \quad y = t^2, \quad z = t^3 \]

we obtain

\[ dx = dt, \quad dy = 2t \, dt, \quad dz = 3t^2 \, dt \]

\therefore

\[ \vec{F} \cdot \vec{dr} = (3t^2 + 6t^2) \, dt - (14t^3) \, 2t \, dt + (20t^2) \, 3t^2 \, dt \]

i.e.,

\[ \vec{F} \cdot \vec{dr} = (9t^2 - 28t^5 + 60t^9) \, dt \quad ; \quad 0 \leq t \leq 1 \]

\therefore \ t \ varies \ from \ 0 \ to \ 1

\[ \int_C \vec{F} \cdot \vec{dr} = \int_0^1 (9t^2 - 28t^5 + 60t^9) \, dt \]

\[ = \left[ \frac{9t^3}{3} - \frac{28t^6}{6} + \frac{60t^{10}}{10} \right]_0^1 \]

\[ = 3 - 4 + 6 \]

\[ = 3 \]

3. Evaluate: \[ \int_C \vec{F} \cdot \vec{dr} \] where \[ \vec{F} = yz \vec{i} + zx \vec{j} + xy \vec{k} \] and \( C \) is given by \[ r = t \vec{i} + t^2 \vec{j} + t^3 \vec{k} \]; \( 0 \leq t \leq 1 \).

Solution

\[ \vec{F} = yz \vec{i} + zx \vec{j} + xy \vec{k} \]

\[ \vec{r} = t \vec{i} + t^2 \vec{j} + t^3 \vec{k} \]

\therefore

\[ \frac{\vec{dr}}{dt} = dt \vec{i} + 2t \, dt \vec{j} + 3t^2 \, dt \vec{k} \]

\therefore

\[ \vec{F} \cdot \vec{dr} = yz \, dt + zx \times 2t \, dt + xy \times 3t^2 \, dt \]

Here

\[ \vec{r} = t \vec{i} + t^2 \vec{j} + t^3 \vec{k} = x \vec{i} + y \vec{j} + z \vec{k} \]

\therefore

\[ x = t, \quad y = t^2, \quad z = t^3 \]

\therefore

\[ \vec{F} \cdot \vec{dr} = t^5 \, dt + 2t^5 \, dt + 3t^5 \, dt \]

\[ = (t^5 + 2t^5 + 3t^5) \, dt \]

\therefore

\[ \vec{F} \cdot \vec{dr} = 6t^5 \, dt \]

\therefore \ t \ varies \ from \ 0 \ to \ 1

\[ \int_C \vec{F} \cdot \vec{dr} = \int_0^1 6t^5 \, dt \]

\[ = \left[ \frac{6t^6}{6} \right]_0^1 \]

\[ = [t^6]_0^1 = 1 - 0 \]

\[ = 1. \]
INTEGRAL THEOREM

Green’s Theorem in a Plane

This theorem gives the relation between the plane, surface and the line integrals.

Statement. If $R$ is a closed region in the $xy$-plane bounded by a simple closed curve $C$ and $M(x, y)$ and $N(x, y)$ are continuous functions having the partial derivatives in $R$ then

$$\int_C M \, dx + N \, dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy.$$

Surface integral and Volume integral

Surface Integral

An integral evaluated over a surface is called a surface integral. Consider a surface $S$ and a point $P$ on it. Let $\vec{A} = A_x \, i + A_y \, j + A_z \, k$ be a vector function of $x, y, z$ defined and continuous over $S$.

In $\hat{n}$ is the unit outward normal to the surface $S$ and $P$ then the integral of the normal component of $\vec{A}$ at $P$ (i.e., $\vec{A} \cdot \hat{n}$) over the surface $S$ is called the surface integral written as

$$\iint_S \vec{A} \cdot \hat{n} \, ds$$

where $ds$ is the small element area. To evaluate integral we have to find the double integral over the orthogonal projection of the surface on one of the coordinate planes.

Suppose $R$ is the orthogonal projection of $S$ on the $XOY$ plane and $\hat{n}$ is the unit outward normal to $S$ then it should be noted that $\hat{n} \cdot \hat{k} \, ds$ ($\hat{k}$ being the unit vector along $z$-axis) is the projection of the vectorial area element $\hat{n} \, ds$ on the $XOY$ plane and this projection is equal to $dx \, dy$ which being the area element in the $XOY$ plane. That is to say that $\hat{n} \cdot \hat{k} \, ds = dx \, dy$. Similarly, we can argue to state that $\hat{n} \cdot \hat{j} \, ds = dz \, dx$ and $\hat{n} \cdot \hat{i} \, ds = dy \, dz$. All these three results hold good if we write $\hat{n} \, ds = dy \, dz \, i + dz \, dx \, j + dx \, dy \, k.$
Sometimes we also write
\[ dS = \hat{n} \, ds = \sum dy \, dz \hat{i} \]

**Volume Integral**

If \( V \) is the volume bounded by a surface and if \( F(x, y, z) \) is a single valued function defined over \( V \) then the volume integral of \( F(x, y, z) \) over \( V \) is given by \( \iiint_V F \, dv \). If the volume is divided into sub-elements having sides \( dx, dy, dz \) then the volume integral is given by the triple integral \( \iiint_S F(x, y, z) \, dx \, dy \, dz \) which can be evaluated by choosing appropriate limits for \( x, y, z \).

**Stoke’s Theorem**

**Statement.** If \( S \) is a surface bounded by a simple closed curve \( C \) and if \( \vec{F} \) is any continuously differentiable vector function then
\[ \oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl} \vec{F} \cdot \hat{n} \, ds = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds \]

**Gauss Divergence Theorem**

**Statement.** If \( V \) is the volume bounded by a surface \( S \) and \( \vec{F} \) is a continuously differentiable vector function then
\[ \iiint_V \text{div} \vec{F} \, dV = \iiint_S \vec{F} \cdot \hat{n} \, dS \]
where \( \hat{n} \) is the positive unit vector outward drawn normal to \( S \).
1. Verify Green’s theorem in the plane for \( \int_C \left( (3x^2 - 8y^2) \, dx + (4y - 6xy) \, dy \right) \) where \( C \) is the boundary for the region enclosed by the parabola \( y^2 = x \) and \( x^2 = y \).

**Solution.** We shall find the points of intersection of the parabolas \( y^2 = x \) and \( x^2 = y \).

\[ i.e., \quad y = \sqrt{x} \text{ and } y = x^2 \]

Equating both, we get

\[ \sqrt{x} = x^2 \implies x = x^4 \]

or

\[ x - x^4 = 0 \]

\[ x (1 - x^3) = 0 \]

\[ \therefore \quad x = 0, 1 \]

and hence \( y = 0, 1 \) the points of intersection are \((0, 0)\) and \((1, 1)\).

Let \( M = 3x^2 - 8y^2 \), \( N = 4y - 6xy \)

\[ \frac{\partial M}{\partial y} = -16y \quad \frac{\partial N}{\partial x} = -6y \]

By Green’s theorem,

\[ \int_C M \, dx + N \, dy = \int_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy \]

L.H.S = \( \int_C M \, dx + N \, dy \)

\[ = \int_{OA} (M \, dx + N \, dy) + \int_{AO} (M \, dx + N \, dy) = I_1 + I_2 \]
Along OA:
\[ y = x^2 \quad dy = 2x \, dx, \]
x varies from 0 to 1
\[
I_1 = \int_{x=0}^{1} \left( 3x^2 - 8x^4 \right) \, dx + \left( 4x^2 - 6x^3 \right) \, 2x \, dx
\]
\[
= \int_{x=0}^{1} \left( 3x^2 + 8x^3 - 20x^4 \right) \, dx
\]
\[
= \left[ x^3 + 2x^4 - 4x^5 \right]_0^1 = -1
\]

Along AO:
\[ y^2 = x \Rightarrow dx = 2y \, dy, \]
y varies from 1 to 0
\[
I_2 = \int_{y=1}^{0} \left( 3y^4 - 8y^2 \right) \, 2y \, dy + \left( 4y - 6y^3 \right) \, dy
\]
\[
= \int_{y=1}^{0} \left( 4y - 22y^3 + 6y^5 \right) \, dy
\]
\[
= \left[ 2y^2 - \frac{11}{2} y^4 + \frac{1}{6} y^6 \right]_1^0 = \frac{5}{2}
\]

\[ \text{Hence,} \quad \text{L.H.S.} = I_1 + I_2 = -1 + \frac{5}{2} = \frac{3}{2} \]

Also
\[ \text{R.H.S.} = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy \]
\[
= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (-6y + 16y) \, dy \, dx
\]
\[
= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} 10y \, dy \, dx
\]
\[
= \int_{x=0}^{1} \left[ \frac{10y^2}{2} \right]_{y=x^2}^{\sqrt{x}} \, dx
\]
\[
= 5 \int_{x=0}^{1} \left( x - x^4 \right) \, dx
\]
\[
= 5 \left[ \frac{x^2}{2} - \frac{x^5}{5} \right]_0^1
\]
\[
= 5 \left[ \frac{1}{2} - \frac{1}{5} \right] = \frac{3}{2}
\]

\[ \therefore \quad \text{L.H.S.} = \text{R.H.S.} = \frac{3}{2} \quad \text{Hence verified.} \]
6. Verify Stoke's theorem for the vector field \( \vec{F} = (2x - y) \hat{i} - yz^2 \hat{j} - y^2 \hat{k} \) over the upper half surface of \( x^2 + y^2 + z^2 = 1 \) bounded by its projection on the xy-plane.

Solution

\[
\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \vec{F} \cdot \hat{n} \, dS
\]

(Stoke's theorem)

\( C \) is the circle: \( x^2 + y^2 = 1, \) \( z = 0 \) (xy-plane)

\( i.e., \) \( x = \cos t, \) \( y = \sin t, \) \( z = 0 \)

\( r = xi + yj \) where \( 0 \leq 0 \leq 2\pi \)

\[
\frac{dr}{dt} = dx\hat{i} + dy\hat{j}
\]

where,

\[
\vec{F} = (2x - y) \hat{i} - yz^2 \hat{j} - y^2 \hat{k}
\]

\[
\therefore \frac{\vec{F} \cdot dr}{\vec{F} \cdot dr} = (2x - y) \, dx
\]

\( \therefore \quad z = 0 \)

L.H.S. = \[
\int_C \vec{F} \cdot \frac{dr}{C} = \int_C (2x - y) \, dx
\]

\[
= \int_0^{2\pi} (2\cos t - \sin t) (-\sin t) \, dt
\]

\[
= \int_0^{2\pi} (2\cos t \sin^2 t) \, dt
\]

\[
= \int_0^{2\pi} (\cos^2 t - 2\cos t \sin t) \, dt
\]

\[
= \int_0^{2\pi} \left[ \frac{1}{2} \left( 1 - \cos 2t \right) - \sin 2t \right] \, dt
\]

\[
= \left[ \frac{t}{2} - \frac{\sin 2t}{4} + \frac{\cos 2t}{2} \right]_0^{2\pi}
\]

\[
= \left( \frac{1}{2} - \frac{1}{2} \right) + (\pi - 0) = \pi
\]

Hence,

\[
\vec{F} \cdot dr = \pi \quad \text{...(1)}
\]

Also,

\[
\text{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix}
\]

\[
= \hat{i} \left( -2yz + 2yz \right) - \hat{j} \left( 0 \right) + \hat{k} \left( 0 \right)\]

\[
= \hat{i} (0) - \hat{j} (0) + \hat{k} (0) = 0 + 0 + 0 = 0
\]
\[ \mathbf{F} = \vec{k} \]
\[ \dot{\mathbf{F}} = \dot{n}dS = dydz \mathbf{i} + dzdx \mathbf{j} + dx dy \mathbf{k} \]

Hence,
\[ \text{R.H.S.} = \iint_{S} \text{curl} \mathbf{F} \cdot \dot{n}dS = \iint_{D} dx dy \]
\[ = \pi \quad \ldots (2) \]

\[ \therefore \iint_{D} dx dy \text{ represents the area of the circle } x^2 + y^2 = 1 \text{ which is } \pi. \]
Thus, from (1) and (2) we conclude that the theorem is verified.

8. Using divergence theorem, evaluate \[ \iiint_{V} \text{div} \mathbf{F} \, dV \]
where \( \mathbf{F} = 4xz \mathbf{i} - y^2 \mathbf{j} + yz \mathbf{k} \) and \( S \) is the surface of the cube bounded by \( x = 0, x = 1, y = 0, y = 1, z = 0, z = 1 \).

Solution. We have divergence theorem:
\[ \iiint_{V} \text{div} \mathbf{F} \, dV = \iint_{S} \mathbf{F} \cdot \dot{n}dS \]

Now \[ \text{div} \mathbf{F} = \nabla \cdot \mathbf{F} \]
\[ = \left( \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (4xz - y^2 j + yzk) \]
\[ = \frac{\partial}{\partial x} (4xz) + \frac{\partial}{\partial y} (-y^2) + \frac{\partial}{\partial z} (yz) \]
\[ = 4z - 2y + y \]
\[ = 4z - y \]

Hence, by divergence theorem, we have
\[ \iint_{S} \mathbf{F} \cdot \dot{n}dS = \iiint_{V} \text{div} \mathbf{F} \cdot dV \]
\[ = \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} (4z - y) \, dz \, dy \, dx \]
\[ = \int_{x=0}^{1} \int_{y=0}^{1} \left[ 2z^2 - yz \right]_{z=0}^{1} \, dy \, dx \]
\[ = \int_{x=0}^{1} \int_{y=0}^{1} (2 - y) \, dy \, dx \]
\[ = \int_{x=0}^{1} \left[ 2y - \frac{y^2}{2} \right]_{y=0}^{1} \, dx \]
\[ = \frac{2}{3} \]
4. Evaluate \( \int_C xy \, dx + xy^2 \, dy \) by Stoke’s theorem where \( C \) is the square in the \( x - y \) plane with vertices \((1, 0), (-1, 0), (0, 1), (0, -1)\).

**Solution.** We have Stoke’s theorem

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{F} \cdot d\mathbf{s}
\]

From the given integral it is evident that

\[
\mathbf{F} = xy \mathbf{i} + xy^2 \mathbf{j}
\]

since,

\[
\frac{d\mathbf{r}}{dt} = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}
\]

Hence,

\[
\int_C xy \, dx + xy^2 \, dy = \int_C \mathbf{F} \cdot d\mathbf{r}
\]

Which is to be evaluated by applying Stoke’s theorem.

Now,

\[
\text{curl} \mathbf{F} = \Delta \times \mathbf{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
xy & xy^2 & 0
\end{vmatrix}
\]

i.e.,

\[
\text{Curl} \mathbf{F} = (y^2 - x) \hat{k}, \text{on expanding the determinant}
\]

Further

\[
\frac{d\mathbf{s}}{ds} = dy \, dz \hat{i} + dz \, dx \hat{j} + dx \, dy \hat{k}
\]

\[
\therefore \iint_S \text{curl} \mathbf{F} \cdot d\mathbf{s} = \iint_S (y^2 - x) \, dx \, dy
\]

It can be clearly seen from the figure that \(-1 \leq x \leq 1\) and \(-1 \leq y \leq 1\)

Now,

\[
\iint_S \text{curl} \mathbf{F} \cdot d\mathbf{s} = \int_{x=-1}^{x=1} \int_{y=-1}^{y=1} (y^2 - x) \, dy \, dx
\]

\[
= \int_{x=-1}^{x=1} \left[ \frac{y^3}{3} - xy \right]_{y=-1}^{y=1} \, dx
\]

\[
= \int_{x=-1}^{x=1} \left[ \left( \frac{1}{3} + 1 \right) - x(1+1) \right] \, dx = \int_{x=-1}^{x=1} \left( \frac{2}{3} - 2x \right) \, dx
\]

\[
= \left[ \frac{2}{3}x - x^2 \right]_{x=-1}^{x=1} = \frac{4}{3}
\]

Thus,

\[
\int_C xy \, dx + xy^2 \, dy = \frac{4}{3}
\]
UNIT VII

LAPLACE TRANSFORMS - I

INTRODUCTION

- Laplace transform is an integral transform employed in solving physical problems.
- Many physical problems when analysed assumes the form of a differential equation subjected to a set of initial conditions or boundary conditions.
- By initial conditions we mean that the conditions on the dependent variable are specified at a single value of the independent variable.
- If the conditions of the dependent variable are specified at two different values of the independent variable, the conditions are called boundary conditions.
- The problem with initial conditions is referred to as the Initial value problem.
- The problem with boundary conditions is referred to as the Boundary value problem.

Example 1: The problem of solving the equation \( \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x \) with conditions \( y(0) = y'(0) = 1 \) is an initial value problem.

Example 2: The problem of solving the equation \( 3 \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + y = \cos x \) with \( y(1)=1, y(2)=3 \) is called Boundary value problem.

Laplace transform is essentially employed to solve initial value problems. This technique is of great utility in applications dealing with mechanical systems and electric circuits. Besides the technique may also be employed to find certain integral values also. The transform is named after the French Mathematician P.S. de’ Laplace (1749 – 1827).

The subject is divided into the following sub topics.
**Definition:**

Let \( f(t) \) be a real-valued function defined for all \( t \geq 0 \) and \( s \) be a parameter, real or complex. Suppose the integral \( \int_0^\infty e^{-st} f(t) \, dt \) exists (converges). Then this integral is called the Laplace transform of \( f(t) \) and is denoted by \( L[f(t)] \).

Thus, \( L[f(t)] = \int_0^\infty e^{-st} f(t) \, dt \) \( (1) \)

We note that the value of the integral on the right hand side of (1) depends on \( s \). Hence \( L[f(t)] \) is a function of \( s \) denoted by \( F(s) \) or \( \tilde{f}(s) \).

Thus, \( \text{L} [f(t)] = F(s) \) \( (2) \)

Consider relation (2). Here \( f(t) \) is called the Inverse Laplace transform of \( F(s) \) and is denoted by \( L^{-1} [F(s)] \).

Thus, \( L^{-1} [F(s)] = f(t) \) \( (3) \)

Suppose \( f(t) \) is defined as follows:

\[
f(t) = \begin{cases} 
f_1(t), & 0 < t < a \\
f_2(t), & a < t < b \\
f_3(t), & t > b 
\end{cases}
\]

Note that \( f(t) \) is piecewise continuous. The Laplace transform of \( f(t) \) is defined as
\[ L[f(t)] = \int_{0}^{\infty} e^{-st} f(t) \]

\[ = \int_{0}^{a} e^{-st} f_1(t) dt + \int_{a}^{b} e^{-st} f_2(t) dt + \int_{b}^{\infty} e^{-st} f_3(t) dt \]

**NOTE:** In a practical situation, the variable \( t \) represents the time and \( s \) represents frequency.

Hence the Laplace transform converts the time domain into the frequency domain.

**Basic properties**

The following are some basic properties of Laplace transforms:

1. **Linearity property**: For any two functions \( f(t) \) and \( \phi(t) \) (whose Laplace transforms exist) and any two constants \( a \) and \( b \), we have

   \[ L [a f(t) + b \phi(t)] = a L[f(t)] + b L[\phi(t)] \]

   **Proof**: By definition, we have

   \[ L [a f(t) + b \phi(t)] = \int_{0}^{\infty} e^{-st} [a f(t) + b \phi(t)] dt \]

   \[ = a \int_{0}^{\infty} e^{-st} f(t) dt + b \int_{0}^{\infty} e^{-st} \phi(t) dt \]

   \[ = a L[f(t)] + b L[\phi(t)] \]

   This is the desired property.

   In particular, for \( a=b=1 \), we have

   \[ L [ f(t) + \phi(t)] = L [f(t)] + L[\phi(t)] \]

   and for \( a = -b = 1 \), we have \[ L [ f(t) - \phi(t)] = L [f(t)] - L[\phi(t)] \]
2. **Change of scale property**: If \( L[f(t)] = F(s) \), then \( L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right) \), where \( a \) is a positive constant.

**Proof**: By definition, we have

\[
L[f(at)] = \int_0^\infty e^{-st} f(at) dt
\]

Let us set \( at = x \). Then expression (1) becomes,

\[
L[f(at)] = \frac{1}{a} \int_0^\infty e^{-\left(\frac{s}{a}\right)x} f(x) dx
\]

\[
= \frac{1}{a} F\left(\frac{s}{a}\right)
\]

This is the desired property.

3. **Shifting property**: Let \( a \) be any real constant. Then

\[
L[e^{at}f(t)] = F(s-a)
\]

**Proof**: By definition, we have

\[
L[e^{at}f(t)] = \int_0^\infty e^{-st} \int_0^\infty e^{at} f(t) dt dt
\]

\[
= \int_0^\infty e^{-(s-a)t} f(t) dt
\]

\[
= F(s-a)
\]

This is the desired property. Here we note that the Laplace transform of \( e^{at} f(t) \) can be written down directly by changing \( s \) to \( s-a \) in the Laplace transform of \( f(t) \).
LAPLACE TRANSFORMS OF STANDARD FUNCTIONS

1. Let \( a \) be a constant. Then

\[
L[e^{at}] = \int_0^\infty e^{-st} e^{at} \, dt = \int_0^\infty e^{-(s-a)t} \, dt
\]

\[
= \left. \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^\infty = \frac{1}{s-a}, \quad s > a
\]

Thus,

\[
L[e^{at}] = \frac{1}{s-a}
\]

In particular, when \( a = 0 \), we get

\[
L(1) = \frac{1}{s}, \quad s > 0
\]

By inversion formula, we have

\[
L^{-1} \frac{1}{s-a} = e^{at} L^{-1} \frac{1}{s} = e^{at}
\]

2. \( L(\cosh at) = L\left(\frac{e^{at} + e^{-at}}{2}\right) = \frac{1}{2} \int_0^\infty e^{-st} \left( e^{at} + e^{-at} \right) \, dt \]

\[
= \frac{1}{2} \int_0^\infty e^{-(s-a)t} + e^{-(s+a)t} \, dt
\]

Let \( s > |a| \). Then,

\[
L(\cosh at) = \frac{1}{2} \left[ \frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)} \right]_0^\infty = \frac{s}{s^2 - a^2}
\]
Thus, \[ L (\cosh at) = \frac{s}{s^2 - a^2}, \quad s > |a| \]

and so

\[ L^{-1}\left(\frac{s}{s^2 - a^2}\right) = \cosh at \]

3. \[ L (\sinh at) = \frac{\frac{e^{at} - e^{-at}}{2}}{s^2 - a^2} = \frac{a}{s^2 - a^2}, \quad s > |a| \]

Thus,

\[ L (\sinh at) = \frac{a}{s^2 - a^2}, \quad s > |a| \]

and so,

\[ L^{-1}\left(\frac{1}{s^2 - a^2}\right) = \frac{\sinh at}{a} \]

4. \[ L (\sin at) = \int_0^\infty e^{-st} \sin at \, dt \]

Here we suppose that \( s > 0 \) and then integrate by using the formula

\[ \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} \left( \sin bx - b \cos bx \right) \]

Thus,

\[ L (\sinh at) = \frac{a}{s^2 + a^2}, \quad s > 0 \]

and so

\[ L^{-1}\left(\frac{1}{s^2 + a^2}\right) = \frac{\sinh at}{a} \]
5. \( L(\cos at) = \int_0^\infty e^{-st} \cos at \, dt \)

Here we suppose that \( s > 0 \) and integrate by using the formula

\[
\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} \left[ \cos bx + b \sin bx \right]
\]

Thus, \( L(\cos at) = \frac{s}{s^2 + a^2}, \quad s > 0 \)

and so \( L^{-1} \frac{s}{s^2 + a^2} = \cos at \)

6. Let \( n \) be a constant, which is a non-negative real number or a negative non-integer. Then

\[
L(t^n) = \int_0^\infty e^{-st} t^n \, dt
\]

Let \( s > 0 \) and set \( st = x \), then

\[
L(t^n) = \int_0^\infty e^{-x} \left( \frac{x}{s} \right)^n \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n \, dx
\]

The integral \( \int_0^\infty e^{-x} x^n \, dx \) is called gamma function of \( (n+1) \) denoted by \( \Gamma(n+1) \).

Thus \( L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} \)

In particular, if \( n \) is a non-negative integer then \( \Gamma(n+1) = n! \). Hence
\[ L(t^n) = \frac{n!}{s^{n+1}} \]

and so

\[ L^{-1} \frac{1}{s^{n+1}} = \frac{t^n}{\Gamma(n+1)} \quad \text{or} \quad \frac{t^n}{n!} \quad \text{as the case may be} \]

**Application of shifting property:**

The shifting property is

If \( L f(t) = F(s) \), then \( L [e^{at}f(t)] = F(s-a) \)

Application of this property leads to the following results:

1. \[ L(e^{at} \cosh bt) = \frac{s-a}{(s-a)^2 - b^2} \]
   
   Thus,
   
   \[ L(e^{at} \cosh bt) = \frac{s-a}{(s-a)^2 - b^2} \]

   and
   
   \[ L^{-1} \frac{s-a}{(s-a)^2 - b^2} = e^{at} \cosh bt \]

   2. \[ L(e^{at} \sinh bt) = \frac{a}{(s-a)^2 - b^2} \]

   and
   
   \[ L^{-1} \frac{1}{(s-a)^2 - b^2} = e^{at} \sinh bt \]
3. \( L(e^{at} \cos bt) = \frac{s-a}{(s-a)^2 + b^2} \)

and

\[ L^{-1} \frac{s-a}{(s-a)^2 + b^2} = e^{at} \cos bt \]

4. \( L(e^{at} \sin bt) = \frac{b}{(s-a)^2 - b^2} \)

and

\[ L^{-1} \frac{1}{(s-a)^2 - b^2} = e^{at} \sin bt \]

5. \( L(e^{at} t^n) = \frac{\Gamma(n+1)}{(s-a)^{n+1}} \) or \( \frac{n!}{(s-a)^{n+1}} \) as the case may be

Hence

\[ L^{-1} \frac{1}{(s-a)^{n+1}} = e^{at} t^n \] or \( \frac{n!}{(s-a)^{n+1}} \) as the case may be

**Examples :-**

1. Find \( L[f(t)] \) given \( f(t) = \begin{cases} t, & 0 < t < 3 \\ 4, & t > 3 \end{cases} \)

Here

\[ L[f(t)] = \int_0^\infty e^{-st} f(t) dt = \int_0^3 e^{-st} dt + \int_3^\infty 4e^{-st} dt \]

Integrating the terms on the RHS, we get
\[ L[f(t)] = \frac{1}{s} e^{-3s} + \frac{1}{s^2} (1 - e^{-3s}) \]

This is the desired result.

2. Find \( L[f(t)] \) given \( L[f(t)] = \sin 2t, \quad 0 < t \leq \pi \)

Here

\[
L[f(t)] = \int_0^\pi e^{-st} f(t) \, dt + \int_\pi^\infty e^{-st} f(t) \, dt = \int_0^\pi e^{-st} \sin 2t \, dt
\]

\[
= \left. \left[ \frac{e^{-st}}{s^2 + 4} \sin 2t - \frac{s}{s^2 + 4} \cos 2t \right] \right|_0^\pi = \frac{2}{s^2 + 4} \left[ e^{-st} \right]_0^\pi
\]

This is the desired result.

3. Evaluate: (i) \( L(\sin 3t \sin 4t) \)
   (ii) \( L(\cos^2 4t) \)
   (iii) \( L(\sin^3 2t) \)

(i) Here \( L(\sin 3t \sin 4t) = L \left[ \frac{1}{2} (\cos t - \cos 7t) \right] \)

\[
= \frac{1}{2} \left[ L(\cos t) - L(\cos 7t) \right] \text{ by using linearity property}
\]

\[
= \frac{1}{2} \left[ \frac{s}{s^2 + 1} - \frac{s}{s^2 + 49} \right] = \frac{24s}{(s^2 + 1)(s^2 + 49)}
\]

(ii) Here

\[
L(\cos^2 4t) = L \left[ \frac{1}{2} (1 + \cos 8t) \right] = \frac{1}{2} \left[ \frac{1}{s} + \frac{s}{s^2 + 64} \right]
\]
(iii) We have
\[ \sin^3 \theta = \frac{1}{4} \left( \sin \theta - \sin 3\theta \right) \]

For \( \theta = 2t \), we get
\[ \sin^3 2t = \frac{1}{4} \left( \sin 2t - \sin 6t \right) \]

so that
\[
L(\sin^3 2t) = \frac{1}{4} \left[ \frac{6}{s^2 + 4} - \frac{6}{s^2 + 36} \right] = \frac{48}{(s^2 + 4)(s^2 + 36)}
\]

This is the desired result.

4. Find \( L(\cos t \cos 2t \cos 3t) \)

Here \( \cos 2t \cos 3t = \frac{1}{2} [\cos 5t + \cos t] \)

so that
\[
\cos t \cos 2t \cos 3t = \frac{1}{2} [\cos 5t \cos t + \cos^2 t]
\]
\[ = \frac{1}{4} [\cos 6t + \cos 4t + 1 + \cos 2t] \]

Thus \( L(\cos t \cos 2t \cos 3t) = \frac{1}{4} \left[ \frac{s}{s^2 + 36} + \frac{s}{s^2 + 16} + \frac{1}{s} + \frac{s}{s^2 + 4} \right] \)

5. Find \( L(\cosh^2 2t) \)

We have
\[ \cosh^2 \theta = \frac{1 + \cosh 2\theta}{2} \]

For \( \theta = 2t \), we get
\[
cosh^2 2t = \frac{1 + \cosh 4t}{2}
\]

Thus,

\[
L(\cosh^2 2t) = \frac{1}{2} \left[ \frac{1}{s} + \frac{s}{s^2 - 16} \right]
\]

6. Evaluate  
(i) \(L(\sqrt{t})\)  
(ii) \(L\left(\frac{1}{\sqrt{t}}\right)\)  
(iii) \(L(t^{3/2})\)

We have \(L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}\)

(i) For \(n = \frac{1}{2}\), we get

\[
L(t^{1/2}) = \frac{\Gamma(\frac{1}{2} + 1)}{s^{3/2}}
\]

Since \(\Gamma(n+1) = n\Gamma(n)\), we have \(\Gamma\left(\frac{1}{2} + 1\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}\)

Thus, \(L(\sqrt{t}) = \frac{\sqrt{\pi}}{2s^{3/2}}\)

(ii) For \(n = -\frac{1}{2}\), we get

\[
L(t^{-1/2}) = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}
\]

(iii) For \(n = -\frac{3}{2}\), we get

\[
L(t^{-3/2}) = \frac{\Gamma\left(-\frac{1}{2}\right)}{s^{3/2}} = \frac{-2\sqrt{\pi}}{s^{3/2}} = -2\sqrt{\pi s}
\]
7. Evaluate: (i) $L(t^2)$ (ii) $L(t^3)$

We have,

\[ L(t^n) = \frac{n!}{s^{n+1}} \]

(i) For $n = 2$, we get

\[ L(t^2) = \frac{2!}{s^3} = \frac{2}{s^3} \]

(ii) For $n=3$, we get

\[ L(t^3) = \frac{3!}{s^4} = \frac{6}{s^4} \]

8. Find $L[e^{-3t}(2\cos5t - 3\sin5t)]$

Given

\[ = 2L(e^{-3t}\cos5t) - 3L(e^{-3t}\sin5t) \]

\[ = 2\frac{\omega + 3}{(s+3)^2 + 25} - \frac{15}{(s+3)^2 + 25}, \text{ by using shifting property} \]

\[ = \frac{2s - 9}{s^2 + 6s + 34}, \text{ on simplification} \]

9. Find $L[\cos at \sin at]$

Here

\[ L[\cos at \sin at] = \frac{L[\cos at] + e^{-at}L[\sin at]}{2} \]

\[ = \frac{1}{2} \left[ \frac{a}{(s-a)^2 + a^2} + \frac{a}{(s+a)^2 + a^2} \right] \]

\[ = \frac{a(s^2 + 2a^2)}{[(s-a)^2 + a^2][(s+a)^2 + a^2]}, \text{ on simplification} \]


10. Find \( L(\cos t \sin^3 2t) \)

Given

\[
L \left[ \frac{e^t + e^{-t}}{2} \left( \frac{3\sin 2t - \sin 6t}{4} \right) \right]
\]

\[= \frac{1}{8} \left[ L(e^t \sin 2t) - L(e^{-t} \sin 2t) + 3L(e^t \sin 2t) - L(e^{-t} \sin 6t) - L(e^t \sin 6t) \right]
\]

\[= \frac{1}{8} \left[ \frac{6}{(s-1)^2 + 4} - \frac{6}{(s-1)^2 + 36} + \frac{6}{(s+1)^2 + 4} - \frac{6}{(s+1)^2 + 36} \right]
\]

\[= \frac{3}{4} \left[ \frac{1}{(s-1)^2 + 4} - \frac{1}{(s-1)^2 + 36} + \frac{1}{(s+1)^2 + 24} - \frac{1}{(s+1)^2 + 36} \right]
\]

11. Find \( L(e^{-4t}t^{-5/2}) \)

We have

\[ L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} \quad \text{Put } n = -5/2. \text{ Hence}
\]

\[ L(t^{-5/2}) = \frac{\Gamma(-3/2)}{s^{-3/2}} - \frac{4\sqrt{\pi}}{3s^{-3/2}} \quad \text{Change } s \text{ to } s+4.
\]

Therefore, \( L(e^{-4t}t^{-5/2}) = \frac{4\sqrt{\pi}}{3(s+4)^{-3/2}} \)

**Transform of \( t^n f(t) \)**

Here we suppose that \( n \) is a positive integer. By definition, we have

\[ F(s) = \int_0^\infty e^{-st} f(t)dt \]
Differentiating ‘n’ times on both sides w.r.t. s, we get

\[ \frac{d^n}{ds^n} F(s) = \frac{\partial^n}{\partial s^n} \int_0^\infty e^{-st} f(t) \, dt \]

Performing differentiation under the integral sign, we get

\[ \frac{d^n}{ds^n} F(s) = \int_0^\infty (-t)^n e^{-st} f(t) \, dt \]

Multiplying on both sides by \((-1)^n\), we get

\[ (-1)^n \frac{d^n}{ds^n} F(s) = \int_0^\infty (t^n f(t)) e^{-st} \, dt = L[t^n f(t)], \text{ by definition} \]

Thus,

\[ L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s) \]

This is the transform of \(t^n f(t)\).

Also,

\[ L^{-1} \left[ \frac{d^n}{ds^n} F(s) \right] = (-1)^n t^n f(t) \]

In particular, we have

\[ L[t f(t)] = - \frac{d}{ds} F(s), \text{ for } n=1 \]

\[ L \left[ t^2 f(t) \right] = \frac{d^2}{ds^2} F(s), \text{ for } n=2, \text{ etc.} \]

Also,

\[ L^{-1} \left[ \frac{d}{ds} F(s) \right] = -tf(t) \text{ and} \]

\[ L^{-1} \left[ \frac{d^2}{ds^2} F(s) \right] = t^2 f(t) \]
Transform of \( \frac{f(t)}{t} \)

We have, \( F(s) = \int_{0}^{\infty} e^{-st} f(t) \, dt \)

Therefore,

\[
\int_{s}^{\infty} F(s) \, ds = \int_{s}^{\infty} \left[ \int_{0}^{\infty} e^{-st} f(t) \, dt \right] \, ds
\]

\[
= \int_{0}^{\infty} f(t) \left[ \int_{s}^{\infty} e^{-st} \, ds \right] \, dt
\]

\[
= \int_{0}^{\infty} f(t) \left[ \frac{e^{-st}}{-t} \right]_{s}^{\infty} \, dt
\]

\[
= \int_{0}^{\infty} e^{-st} \left[ \frac{f(t)}{t} \right] \, dt = L\left( \frac{f(t)}{t} \right)
\]

Thus,

\[
L\left( \frac{f(t)}{t} \right) = \int_{s}^{\infty} F(s) \, ds
\]

This is the transform of \( \frac{f(t)}{t} \)

Also, \( L^{-1} \int_{s}^{\infty} F(s) \, ds = \frac{f(t)}{t} \)

**Examples:**

1. Find \( L \left[ te^{-t} \sin(4t) \right] \)

   We have, \( L[e^{-t} \sin(4t)] = \frac{4}{(s+1)^2 + 16} \)

   So that,
\[ L [t e^{-t} \sin 4t] = 4 \left( -\frac{d}{ds} \left( \frac{1}{s^2 + 2s + 17} \right) \right) \]
\[ = \frac{8(s + 1)}{(s^2 + 2s + 17)^2} \]

2. Find \( L (t^2 \sin 3t) \)

We have \( L (\sin 3t) = \frac{3}{s^2 + 9} \)

So that,
\[ L (t^2 \sin 3t) = \frac{d^2}{ds^2} \left( \frac{3}{s^2 + 9} \right) \]
\[ = -6 \frac{d}{ds} \frac{s}{(s^2 + 9)^2} \]
\[ = \frac{18(s^2 - 3)}{(s^2 + 9)^3} \]

3. Find \( L \left( \frac{e^{-t} \sin t}{t} \right) \)

We have
\[ L(e^{-t} \sin t) = \frac{1}{(s + 1)^2 + 1} \]

Hence
\[ L \left( \frac{e^{-t} \sin t}{t} \right) = \int_{0}^{\infty} \frac{ds}{(s + 1)^2 + 1} = \tan^{-1} (s + 1) \bigg|_{0}^{\infty} = \tan^{-1} (s + 1) \bigg|_{0}^{\infty} = \frac{\pi}{2} - \tan^{-1} (s + 1) = \cot^{-1} (s + 1) \]

4. Find \( L \left( \frac{\sin t}{t} \right) \). Using this, evaluate \( L \left( \frac{\sin at}{t} \right) \)

We have \( L (\sin t) = \frac{1}{s^2 + 1} \)

So that
\[ L [f(t)] = L \left( \frac{\sin t}{t} \right) = \int_{s}^{\infty} \frac{ds}{s^2 + 1} = \tan^{-1} s \bigg|_{s}^{\infty} \]
\[ = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s = F(s) \]

Consider

\[ L \left( \frac{\sin at}{t} \right) = a L \left( \frac{\sin at}{at} \right) = a f(at) \]

\[ = a \left[ \frac{1}{a} F \left( \frac{s}{a} \right) \right], \text{ in view of the change of scale property} \]

\[ = \cot^{-1} \left( \frac{s}{a} \right) \]

5. Find \( L \left[ \frac{\cos at - \cos bt}{t} \right] \)

We have \( L \left[ \cos at - \cos bt \right] = \frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} \).

So that

\[ L \left[ \frac{\cos at - \cos bt}{t} \right] = \int_{s}^{\infty} \left[ \frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} \right] ds \]

\[ = \frac{1}{2} \left[ \log \left( \frac{s^2 + a^2}{s^2 + b^2} \right) \right]_{s}^{\infty} \]

\[ = \frac{1}{2} \left[ \lim_{s \to \infty} \log \left( \frac{s^2 + a^2}{s^2 + b^2} \right) - \log \left( \frac{s^2 + a^2}{s^2 + b^2} \right) \right] \]

\[ = \frac{1}{2} \left[ 0 + \log \left( \frac{s^2 + b^2}{s^2 + a^2} \right) \right] \]

\[ = \frac{1}{2} \log \left( \frac{s^2 + b^2}{s^2 + a^2} \right) \]
6. Prove that \( \int_{0}^{\infty} e^{-st} \sin t \, dt = \frac{3}{50} \)

We have

\[
\int_{0}^{\infty} e^{-st} \sin t \, dt = L(t \sin t) = -\frac{d}{ds} L(t) = -\frac{d}{ds} \left[ \frac{1}{s^2 + 1} \right] = \frac{2s}{(s^2 + 1)^2}
\]

Putting \( s = 3 \) in this result, we get

\[
\int_{0}^{\infty} e^{-st} \sin t \, dt = \frac{3}{50}
\]

This is the result as required.

Consider

\[
L[f'(t)] = \int_{0}^{\infty} e^{-st} f'(t) \, dt
\]

\[
= \int_{0}^{\infty} f(t) \frac{e^{-st}}{s} - \int_{0}^{\infty} e^{-st} f(t) \, dt , \text{ by using integration by parts}
\]

\[
= \int_{0}^{\infty} t (e^{-st} f(t) - f(0)) + s L[f(t)]
\]

Thus

\[
L[f'(t)] = s L[f(t)] - f(0)
\]

Similarly,

\[
L[f''(t)] = s^2 L[f(t)] - s f(0) + f'(0)
\]
In general, we have

\[ Lf^n(t) = s^n Lf(t) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{n-1}(0) \]

**Transform of \( \int_0^t f(t) \, dt \)**

Let \( \phi(t) = \int_0^t f(t) \, dt \). Then \( \phi(0) = 0 \) and \( \phi'(t) = f(t) \)

Now, \( \mathcal{L} \phi(t) = \int_0^\infty e^{-st} \phi(t) \, dt \)

\[ = \left[ \phi(t) \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty \phi'(t) \frac{e^{-st}}{-s} \, dt \]

\[ = (0 - 0) - \int_0^\infty f(t) e^{-st} \, dt \]

Thus, \( \mathcal{L} \int_0^t f(t) \, dt = \frac{1}{s} \mathcal{L}[f(t)] \)

Also, \( \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L}[f(t)] \right] = \int_0^t f(t) \, dt \)

**Examples:**

1. By using the Laplace transform of \( \sin at \), find the Laplace transforms of \( \cos at \).

Let \( f(t) = \sin at \), then \( \mathcal{L}[f(t)] = \frac{a}{s^2 + a^2} \)

We note that

\[ f'(t) = a \cos at \]

Taking Laplace transforms, we get

\[ Lf'(t) = L(a \cos at) = aL(\cos at) \]
or \( L(\cos at) = \frac{1}{a} Lf'(t) = \frac{1}{a} \left[ Lf(t) - f(0) \right] \)

\[
= \frac{1}{a} \left[ \frac{sa}{s^2 + a^2} - 0 \right]
\]

Thus

\[
L(\cos at) = \frac{s}{s^2 + a^2}
\]

This is the desired result.

2. Given \( L \left[ 2 \sqrt{\frac{t}{\pi}} \right] = \frac{1}{s^{3/2}} \), show that \( L \left[ \frac{1}{\sqrt{\pi t}} \right] = \frac{1}{\sqrt{s}} \)

Let \( f(t) = 2 \sqrt{\frac{t}{\pi}} \), given \( L[f(t)] = \frac{1}{s^{3/2}} \)

We note that, \( f'(t) = \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2\sqrt{t}} = \frac{1}{\sqrt{\pi t}} \)

Taking Laplace transforms, we get

\[
Lf'(t) = L \left[ \frac{1}{\sqrt{\pi t}} \right]
\]

Hence

\[
L \left[ \frac{1}{\sqrt{\pi t}} \right] = Lf'(t) = sLf(t) - f(0)
\]

\[
= s \left( \frac{1}{s^{3/2}} \right) - 0
\]

Thus \( L \left[ \frac{1}{\sqrt{\pi t}} \right] = \frac{1}{\sqrt{s}} \)

This is the result as required.
3. Find \( L \int_0^t \left( \frac{\cos at - \cos bt}{t} \right) dt \)

Here

\[
L[f(t)] = L\left( \frac{\cos at - \cos bt}{t} \right) = \frac{1}{2} \log \left( \frac{s^2 + b^2}{s^2 + a^2} \right)
\]

Using the result

\[
L \int_0^t f(t) dt = \frac{1}{s} Lf(t)
\]

We get,

\[
L \int_0^t \left( \frac{\cos at - \cos bt}{t} \right) dt = \frac{1}{2s} \log \left( \frac{s^2 + b^2}{s^2 + a^2} \right)
\]

4. Find \( L \int_0^t e^{-t} \sin 4tdt \)

Here

\[
L \left[ e^{-t} \sin 4t \right] = \frac{8(s + 1)}{(s^2 + 2s + 17)^2}
\]

Thus

\[
L \int_0^t e^{-t} \sin 4tdt = \frac{8(s + 1)}{s(s^2 + 2s + 17)^2}
\]
**Laplace Transform of a periodic function**

A function $f(t)$ is said to be a periodic function of period $T > 0$ if $f(t) = f(t + nT)$ where $n=1,2,3,\ldots$. The graph of the periodic function repeats itself in equal intervals.

For example, $\sin t$, $\cos t$ are periodic functions of period $2\pi$ since $\sin (t + 2n\pi) = \sin t$, $\cos(t + 2n\pi) = \cos t$.

The graph of $f(t) = \sin t$ is shown below:

Note that the graph of the function between $0$ and $2\pi$ is the same as that between $2\pi$ and $4\pi$ and so on.

The graph of $f(t) = \cos t$ is shown below:
Note that the graph of the function between 0 and $2\pi$ is the same as that between $2\pi$ and $4\pi$ and so on.

**Formula:** Let $f(t)$ be a periodic function of period $T$. Then

$$Lf(t) = \frac{1}{1 - e^{-ST}} \int_{0}^{T} e^{-st} f(t) dt$$

Proof: By definition, we have

$$L f(t) = \int_{0}^{\infty} e^{-st} f(t) dt = \int_{0}^{\infty} e^{-su} f(u) du$$

$$= \int_{0}^{T} e^{-su} f(u) du + \int_{T}^{2T} e^{-su} f(u) du + \ldots + \int_{nT}^{(n+1)T} e^{-su} f(u) du + \ldots + \infty$$

$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-su} f(u) du$$

Let us set $u = t + nT$, then

$$L f(t) = \sum_{n=0}^{\infty} \int_{t=0}^{T} e^{-s(t+nT)} f(t+nT) dt$$

Here

$$f(t+nT) = f(t)$$

by periodic property

Hence

$$Lf(t) = \sum_{n=0}^{\infty} (e^{-ST})^{n} \int_{0}^{T} e^{-st} f(t) dt$$

$$= \left[ \frac{1}{1 - e^{-ST}} \right]^{T} \int_{0}^{T} e^{-st} f(t) dt$$

identifying the above series as a geometric series.

Thus
\[ L[f(t)] = \left[ \frac{1}{1 - e^{-st}} \right]^{T} \int_{0}^{T} e^{-st} f(t) dt \]

This is the desired result.

**Examples:**

1. For the periodic function \( f(t) \) of period 4, defined by \( f(t) = \begin{cases} 3t, & 0 < t < 2 \\ 6, & 2 < t < 4 \end{cases} \)

find \( L[f(t)] \)

Here, period of \( f(t) = T = 4 \)

We have,

\[ L f(t) = \left[ \frac{1}{1 - e^{-st}} \right]^{4} \int_{0}^{4} e^{-st} f(t) dt \]

\[ = \left[ \frac{1}{1 - e^{-4s}} \right]^{4} \int_{0}^{4} e^{-st} f(t) dt \]

\[ = \frac{1}{1 - e^{-4s}} \left[ \int_{0}^{2} 3te^{-st} dt + \int_{2}^{4} 6e^{-st} dt \right] \]

\[ = \frac{1}{1 - e^{-4s}} \left[ 3 \left( t \left( \frac{e^{-st}}{-s} \right) \right) \bigg|_{0}^{2} - \int_{0}^{2} \left( \frac{e^{-st}}{-s} \right) dt \right] + 6 \left( \frac{e^{-st}}{-s} \right) \bigg|_{2}^{4} \]

\[ = \frac{1}{1 - e^{-4s}} \left[ 3 \left( -e^{-2s} - 2se^{-4s} \right) \right] \]

Thus,

\[ L[f(t)] = \frac{3(1 - e^{-2s} - 2se^{-4s})}{s^2(1 - e^{-4s})} \]
3. A periodic function of period \( \frac{2\pi}{\omega} \) is defined by

\[
f(t) = \begin{cases} 
esin \omega t, & 0 \leq t < \frac{\pi}{\omega} \\
0, & \frac{\pi}{\omega} \leq t \leq \frac{2\pi}{\omega}
\end{cases}
\]

where \( E \) and \( \omega \) are positive constants. Show that

\[
L[f(t)] = \frac{E\omega}{(s^2 + \omega^2)(1 - e^{-\pi/s})}
\]

Here \( T = \frac{2\pi}{\omega} \). Therefore

\[
L[f(t)] = \frac{1}{1 - e^{-s(2\pi/\omega)}} \int_{0}^{\frac{2\pi}{\omega}} e^{-st} f(t) dt
\]

\[
= \frac{1}{1 - e^{-s(2\pi/\omega)}} \int_{0}^{\frac{\pi}{\omega}} E e^{-st} \sin \omega t dt
\]

\[
= \frac{E}{1 - e^{-s(2\pi/\omega)}} \left[ \frac{e^{-st}}{s^2 + \omega^2} - ts \sin \omega t - \omega \cos \omega t \right]_{0}^{\frac{\pi}{\omega}}
\]

\[
= \frac{E}{1 - e^{-s(2\pi/\omega)}} \frac{\omega(e^{-s\pi/\omega} + 1)}{s^2 + \omega^2}
\]

\[
= \frac{E\omega(1 + e^{-s\pi/\omega})}{(1 - e^{-s\pi/\omega})(1 + e^{-s\pi/\omega})(s^2 + \omega^2)}
\]

\[
= \frac{E\omega}{(1 - e^{-s\pi/\omega})(s^2 + \omega^2)}
\]

This is the desired result.
3. A periodic function $f(t)$ of period $2a$, $a > 0$ is defined by

$$f(t) = \begin{cases} 
E, & 0 \leq t \leq a \\
-E, & a < t \leq 2a
\end{cases}$$

show that

$$L[f(t)] = \frac{E}{s} \tanh \left( \frac{as}{2} \right)$$

Here $T = 2a$. Therefore

$$L[f(t)] = \frac{1}{1-e^{-2as}} \int_{0}^{2a} e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{-2as}} \left[ \int_{0}^{a} E e^{-st} dt + \int_{a}^{2a} -E e^{-st} dt \right]$$

$$= \frac{E}{s(1-e^{-2as})} \left[ -e^{-as} \pm (e^{-2as} - e^{-as}) \right]$$

$$= \frac{E}{s(1-e^{-2as})} \left[ -e^{-as} \right]$$

$$= \frac{E(1-e^{-as})^2}{s(1-e^{-as})(1+e^{-as})}$$

$$= \frac{E}{s} \left[ \frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}} \right]$$

$$= \frac{E}{s} \tanh \left( \frac{as}{2} \right)$$

This is the result as desired.
Step Function:

In many Engineering applications, we deal with an important discontinuous function \( H(t-a) \) defined as follows:

\[
H(t-a) = \begin{cases} 
0, & t \leq a \\
1, & t > a 
\end{cases}
\]

where \( a \) is a non-negative constant.

This function is known as the unit step function or the Heaviside function. The function is named after the British electrical engineer Oliver Heaviside. The function is also denoted by \( u(t-a) \). The graph of the function is shown below:

\[ H(t-1) \]

Note that the value of the function suddenly jumps from value zero to the value 1 as \( t \to a \) from the left and retains the value 1 for all \( t > a \). Hence the function \( H(t-a) \) is called the unit step function.

In particular, when \( a = 0 \), the function \( H(t-a) \) becomes \( H(t) \), where

\[
H(t) = \begin{cases} 
0, & t \leq 0 \\
1, & t > 0 
\end{cases}
\]

Transform of step function

By definition, we have \( L[H(t-a)] = \int_0^\infty e^{-st} H(t-a) dt \)

\[
= \int_0^a e^{-st} 0 dt + \int_a^\infty e^{-st} 1 dt
\]

\[
= \frac{e^{-as}}{s}
\]
In particular, we have \( L H(t) = \frac{1}{s} \)

Also,  
\[
L^{-1}\left[ \frac{e^{-as}}{s} \right] = H(t - a) \quad \text{and} \quad L^{-1}\left( \frac{1}{s} \right) = H(t)
\]

**Unit step function (Heaviside function)**

**Statement:** - \( L [f(t-a) H(t-a)] = e^{-as} Lf(t) \)

**Proof:** - We have

\[
L [f(t-a) H(t-a)] = \int_0^\infty f(t-a)H(t-a)e^{-st} \, dt
\]

\[
= \int_a^\infty e^{-st} f(t-a) \, dt
\]

Setting \( t-a = u \), we get

\[
L[f(t-a) H(t-a)] = \int_0^\infty e^{-s(a+u)} f(u) \, du
\]

\[
= e^{-as} L [f(t)]
\]

This is the desired shift theorem.

Also, \( L^{-1} [e^{-as} Lf(t)] = f(t-a) H(t-a) \)

**Examples:**

1. Find \( L [e^{t/2} + \sin(t-2)] H(t-2) \)

   Let 

   \[ f(t-2) = [e^{t/2} + \sin(t-2)] \]

   Then \( f(t) = [e^t + \sin t] \)

   so that \( L f(t) = \frac{1}{s-1} + \frac{1}{s^2+1} \)
By Heaviside shift theorem, we have

\[ L[f(t-2) \, H(t-2)] = e^{-2s} \, Lf(t) \]

Thus,

\[ L[e^{(t-2)} + \sin(t-2)]H(t-2) = e^{-2s} \left[ \frac{1}{s-1} + \frac{1}{s^2 + 1} \right] \]

2. Find \( L(3t^2 + 2t + 3) \, H(t-1) \)

Let

\[ f(t-1) = 3t^2 + 2t + 3 \]

so that

\[ f(t) = 3(t+1)^2 + 2(t+1) + 3 = 3t^2 + 8t + 8 \]

Hence

\[ L[f(t)] = \frac{6}{s^3} + \frac{1}{s^2} + \frac{8}{s} \]

Thus

\[ L \left[ 3t^2 + 2t + 3 \right] H(t-1) = L[f(t-1) \, H(t-1)] \]

\[ = e^{-s} \, L \left[ f(t) \right] \]

\[ = e^{-s} \left[ \frac{6}{s^3} + \frac{8}{s^2} + \frac{8}{s} \right] \]

3. Find \( Le^{-t} \, H(t-2) \)

Let \( f(t-2) = e^{-t} \), so that \( f(t) = e^{-(t+2)} \)
Thus,

\[ L \{f(t)\} = \frac{e^{-2}}{s + 1} \]

By shift theorem, we have

\[ L[ f(t - 2)H(t - 2)] = e^{-2s}L(f(t)) = \frac{e^{-2(s+1)}}{s + 1} \]

Thus

\[ L \left[ \frac{e^{-2(s+1)}}{s + 1} \right] = \frac{e^{-2(s+1)}}{s + 1} \]

4. Let \( f(t) = \begin{cases} f_1(t), & t \leq a \\ f_2(t), & t > a \end{cases} \)

Verify that \( f(t) = f_1(t) + [f_2(t) - f_1(t)]H(t-a) \)

Consider

\[ f_1(t) + [f_2(t) - f_1(t)]H(t-a) = \begin{cases} f_1(t) + f_2(t) - f_1(t), & t > a \\ 0, & t \leq a \end{cases} \]

\[ = \begin{cases} f_2(t), & t > a \\ f_1(t), & t \leq a \end{cases} = f(t), \text{ given} \]

Thus the required result is verified.

5. Express the following functions in terms of unit step function and hence find their Laplace transforms.

1. \( f(t) = \begin{cases} t^2, & 1 < t \leq 2 \\ 4t, & t > 2 \end{cases} \)
Here, \( f(t) = t^2 + (4t-t^2) \, H(t-2) \)

Hence, \( \mathcal{L} \{ f(t) \} = \frac{2}{s^3} + L(4t - t^2)H(t-2) \) \hspace{1cm} (i)

Let \( \phi(t-2) = 4t - t^2 \)

so that \( \phi(t) = 4(t+2) - (t+2)^2 = -t^2 + 4 \)

Now, \( \mathcal{L} \{ \phi(t) \} = -\frac{2}{s^3} + \frac{4}{s} \)

Expression (i) reads as

\[
\mathcal{L} \{ f(t) \} = \frac{2}{s^3} + \mathcal{L} \left[ (t-2)H(t-2) \right] \\
= \frac{2}{s^3} + e^{-2s} \mathcal{L} \{ \phi(t) \} \\
= \frac{2}{s^3} + e^{-2s} \left( \frac{4}{s} - \frac{2}{s^3} \right)
\]

This is the desired result.

2. \( f(t) = \begin{cases} 
\cos(t), & 0 < t < \pi \\
\sin(t), & t > \pi
\end{cases} \)

>> Here \( f(t) = \cos(t) + (\sin(t) - \cos(t))H(t-\pi) \)

Hence,

\[
\mathcal{L} \{ f(t) \} = \frac{s}{s^2 + 1} + \mathcal{L} \{ \sin(t) - \cos(t) \}H(t-\pi) \] \hspace{1cm} (ii)

Let

\( \phi(t-\pi) = \sin(t) - \cos(t) \)

Then
\[ \phi(t) = \sin(t + \pi) - \cos(t + \pi) = -\sin t + \cos t \]

so that

\[ L[\phi(t)] = -\frac{1}{s^2 + 1} + \frac{s}{s^2 + 1} \]

Expression (ii) reads as

\[ L[f(t)] = \frac{s}{s^2 + 1} + L\{\phi(t - \pi)H(t - \pi)\} \]

\[ = \frac{s}{s^2 + 1} + e^{-\pi s} L\phi(t) \]

\[ = \frac{s}{s^2 + 1} + e^{-\pi s} \left[ \frac{s-1}{s^2 + 1} \right] \]

**UNIT IMPULSE FUNCTION**

**Definition:** The unit impulse function denoted by \( \delta(t - a) \) is defined as follows

\[
\delta(t - a) = \lim_{\epsilon \to 0} \delta_{\epsilon}(t - a), \quad a \geq 0
\]

\[= \begin{cases} 
0, & \text{if } t < a \\
\frac{1}{\epsilon}, & \text{if } a < t < a + \epsilon \\
0, & \text{if } t > a + \epsilon 
\end{cases} \quad \ldots (2)
\]

The graph of the function \( \delta_{\epsilon}(t - a) \) is as shown below:

![Graph of the unit impulse function](image)

Fig. 7.2

Laplace transform of the unit impulse function
Consider \[ L \{ \delta_{\varepsilon} (t-a) \} = \int_{0}^{\infty} e^{-st} \delta_{\varepsilon} (t-a) \, dt \]
\[ = \int_{0}^{a} e^{-st} \, dt + \int_{a}^{a+\varepsilon} e^{-st} \frac{1}{\varepsilon} \, dt + \int_{a+\varepsilon}^{\infty} e^{-st} \, dt \]
\[ = \frac{1}{\varepsilon} \int_{a}^{a+\varepsilon} e^{-st} \, dt = \frac{1}{\varepsilon} \left[ \frac{e^{-st}}{-s} \right]_{a}^{a+\varepsilon} \]
\[ = -\frac{1}{\varepsilon s} \left[ e^{-sa}(a+\varepsilon) - e^{-sa} \right] \]
\[ = e^{-sa} \left[ \frac{1 - e^{-\varepsilon s}}{\varepsilon s} \right] \]

Taking the limits on both sides as \( \varepsilon \to 0 \), we get,
\[ \lim_{\varepsilon \to 0} L \{ \delta_{\varepsilon} (t-a) \} = e^{-as} \lim_{\varepsilon \to 0} \left[ \frac{1 - e^{-\varepsilon s}}{\varepsilon s} \right] \]

i.e., \[ L \{ \delta (t-a) \} = e^{-as} \] (Using L’ Hospital Rule)

If \[ a = 0 \text{ then } L \{ \delta (t) \} = 1 \]

1. Find the Laplace transforms of the following functions:

   (1) \((2t-1)u(t-2)\)

   Solution

   (1) Now \[ 2t-1 = 2(t-2) + 3 \]

   \[ \therefore \text{Using Heaviside shift theorem, we get} \]

   \[ L \{ (2t-1)u(t-2) \} = L \{ [2(t-2) + 3]u(t-2) \} \]

   \[ = e^{-2s} L \{ 2t + 3 \} \]

   \[ = e^{-2s} \left( 2L \{ t \} + L \{ 3 \} \right) \]

   \[ = e^{-2s} \left\{ \frac{2}{s^2} + \frac{3}{s} \right\}. \]

   (2) \( t^2 u(t-3)\)

   \[ t^2 = [(t-3) + 3]^2 \]
Then \( L \{t^2 u(t-3)\} = L \{(t-3)^2 + 6(t-3) + 9\} u(t-3) \)

Replacing \(t-3\) by \(t\)

\[ = e^{-3s} L\{t^2 + 6t + 9\} \]

Using Heaviside shift theorem

\[ = e^{-3s} \left\{ \frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s} \right\} . \]

**Find** \( L\left[2\delta(t-1) + 3\delta(t-2) + 4\delta(t+3)\right] \).

**Solution.** We have

\[ 2L \delta(t-1) + 3L \delta(t-2) + 4L \delta(t+3) \]

\[ = 2e^{-s} + 3e^{-2s} + 4e^{3s} . \]

Since \( L \delta(t-a) = e^{-as} \)

**Find** \( L\left[\cosh 3t \delta(t-2)\right] \).

**Solution**

\[ \cosh 3t \delta(t-2) = \frac{1}{2} \left\{ e^{3t} + e^{-3t} \right\} \delta(t-2) \]

\[ L\left[\cosh 3t \delta(t-2)\right] = \frac{1}{2} \left\{ L\left[e^{3t} \delta(t-2)\right] + L\left[e^{-3t} \delta(t-2)\right] \right\} \]

= shifting \( s - 3 \rightarrow s \) \( s + 3 \rightarrow s \)

\[ = \frac{1}{2} \left\{ L\left[\delta(t-2)\right]_{s \rightarrow s-3} + L\left[\delta(t-2)\right]_{s \rightarrow s+3} \right\} \]

\[ = \frac{1}{2} \left\{ \left(e^{-2s}\right)_{s \rightarrow s-3} + \left(e^{-2s}\right)_{s \rightarrow s+3} \right\} \]

\[ = \frac{1}{2} \left\{ e^{-2(s-3)} + e^{-2(s+3)} \right\} \]

\[ = \frac{e^{-2s}}{2} \left\{ e^6 + e^{-6} \right\} \]

\( L\left[\cosh 3t \delta(t-2)\right] = \cosh 6 e^{-2s} \)
**UNIT VIII**

**LAPLACE TRANSFORMS – 2**

**Introduction:**

Let $L [f(t)] = F(s)$. Then $f(t)$ is defined as the inverse Laplace transform of $F(s)$ and is denoted by $L^{-1} F(s)$. Thus $L^{-1} [F(s)] = f(t)$.

**Linearity Property**

Let $L^{-1} [F(s)] = f(t)$ and $L^{-1} [G(s) = g(t)]$ and $a$ and $b$ be any two constants. Then $L^{-1} [a F(s) + b G(s)] = a L^{-1} [F(s)] + b L^{-1} [G(s)]$

**Table of Inverse Laplace Transforms**

<table>
<thead>
<tr>
<th>$F(s)$</th>
<th>$f(t) = L^{-1} F(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{s}$, $s &gt; 0$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\frac{1}{s-a}$, $s &gt; a$</td>
<td>$e^{at}$</td>
</tr>
<tr>
<td>$\frac{s}{s^2 + a^2}$, $s &gt; 0$</td>
<td>$\text{Cos at}$</td>
</tr>
<tr>
<td>$\frac{1}{s^2 + a^2}$, $s &gt; 0$</td>
<td>$\frac{\text{Sin at}}{a}$</td>
</tr>
<tr>
<td>$\frac{1}{s^2 - a^2}$, $s &gt;</td>
<td>a</td>
</tr>
<tr>
<td>$\frac{s}{s^2 - a^2}$, $s &gt;</td>
<td>a</td>
</tr>
<tr>
<td>$\frac{1}{s^{n+1}}$, $s &gt; 0$</td>
<td>$\frac{t^n}{n!}$</td>
</tr>
<tr>
<td>$n = 0, 1, 2, 3, \ldots$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{s^{n+1}}$, $s &gt; 0$</td>
<td>$\frac{t^n}{\Gamma(n+1)}$</td>
</tr>
<tr>
<td>$n &gt; -1$</td>
<td></td>
</tr>
</tbody>
</table>
Examples

1. Find the inverse Laplace transforms of the following:

(i) \( \frac{1}{2s-5} \) \hspace{2cm} (ii) \( \frac{s+b}{s^2+a^2} \) \hspace{2cm} (iii) \( \frac{2s-5}{4s^2+25} + \frac{4s-9}{9-s^2} \)

Here

(i) \( L^{-1} \left[ \frac{1}{2s-5} \right] = \frac{1}{2} L^{-1} \left[ \frac{1}{s-\frac{5}{2}} \right] = \frac{1}{2} e^{\frac{5t}{2}} \)

(ii) \( L^{-1} \left[ \frac{s+b}{s^2+a^2} \right] = L^{-1} \left[ \frac{s}{s^2+a^2} \right] + b L^{-1} \left[ \frac{1}{s^2+a^2} \right] = \cos at + \frac{b}{a} \sin at \)

(iii) \( L^{-1} \left[ \frac{2s-5}{4s^2+25} + \frac{4s-9}{9-s^2} \right] = \frac{2}{4} L^{-1} \left[ \frac{s-\frac{5}{2}}{s^2+\frac{25}{4}} \right] - 4L^{-1} \left[ \frac{s-\frac{9}{2}}{s^2-9} \right] \)

\[ = \frac{1}{2} \left[ \cos \frac{5t}{2} - \sin \frac{5t}{2} \right] - 4 \left[ \cos 3t - \frac{3}{2} \sin 3t \right] \]

Evaluation of \( L^{-1} F(s-a) \)

We have, if \( L \left[ f(t) \right] = F(s) \), then \( L[e^{at} f(t)] = F(s-a) \), and so

\( L^{-1} \left[ F(s-a) \right] = e^{at} f(t) = e^{at} L^{-1} \left[ F(s) \right] \)

Examples

1. Evaluate : \( L^{-1} \left[ \frac{3s+1}{\xi+1^3} \right] \)

Given = \( L^{-1} \left[ \frac{3(\xi+1)+1}{\xi+1^2} \right] = 3 L^{-1} \left[ \frac{1}{\xi+1^2} \right] - 2 L^{-1} \left[ \frac{1}{\xi+1^3} \right] \)
\[= 3e^{-t} L^{-1}\left[\frac{1}{s^3}\right] - 2e^{-t} L^{-1}\left[\frac{1}{s^4}\right] \]

Using the formula

\[L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!} \quad \text{and taking } n = 2 \text{ and } 3, \text{ we get} \]

\[\text{Given} = \frac{3e^{-t}t^2}{2} - \frac{e^{-t}t^3}{3} \]

2. Evaluate: \(L^{-1}\left[\frac{s+2}{s^2 - 2s + 5}\right]\)

\[\text{Given} = L^{-1}\left[\frac{s+2}{s^2 - 2s + 5}\right] = L^{-1}\left[\frac{s+2}{s^2 - 2s + 5}\right] = \frac{s+2}{s^2 - 2s + 5} = L^{-1}\left[\frac{1}{s^2 - 2s + 5}\right], \quad 3L^{-1}\left[\frac{1}{s^2 + 4}\right] \]

\[= e^t L^{-1}\left[\frac{s}{s^2 + 4}\right] + 3 e^t L^{-1}\left[\frac{1}{s^2 + 4}\right] \]

\[= e^t \cos 2t + \frac{3}{2} e^t \sin 2t \]

Evaluate: \(L^{-1}\left[\frac{2s+1}{s^3 + 3s + 1}\right]\)

\[\text{Given} = 2L^{-1}\left[\frac{s+1}{s^3 + 3s + 1}\right] = 2L^{-1}\left[\frac{s+1}{s^3 + 3s + 1}\right] = 2L^{-1}\left[\frac{s+1}{s^3 + 3s + 1}\right] = L^{-1}\left[\frac{1}{s^3 + 3s + 1}\right] \]

\[= 2\left[\frac{e^{-\frac{3t}{2}}}{s^2 - \frac{5}{4}}\right] - e^{-\frac{3t}{2}} \left[\frac{1}{s^2 - \frac{5}{4}}\right] \]

\[= 2e^{-\frac{3t}{2}} \left[\cos \frac{\sqrt{5}}{2} t - \frac{2}{\sqrt{5}} \sin \frac{\sqrt{5}}{2} t\right] \]
4. Evaluate: \[ L^{-1} \left[ \frac{2s^2 + 5s - 4}{s^3 + s^2 - 2s} \right] \]

we have

\[ \frac{2s^2 + 5s - 4}{s^3 + s^2 - 2s} = \frac{2s^2 + 5s - 4}{s(s^2 + s - 2)} \]
\[ = \frac{2s^2 + 5s - 4}{s(s + 2)(s - 1)} \]
\[ = \frac{A}{s} + \frac{B}{s + 2} + \frac{C}{s - 1} \]

Then \[ 2s^2 + 5s - 4 = A(s + 2)(s - 1) + B(s - 1) + C(s + 2) \]

For \( s = 0 \), we get \( A = 2 \), for \( s = 1 \), we get \( C = 1 \) and for \( s = -2 \), we get \( B = -1 \). Using these values in (1), we get

\[ \frac{2s^2 + 5s - 4}{s^3 + s^2 - 2s} = \frac{2}{s} - \frac{1}{s + 2} + \frac{1}{s - 1} \]

Hence

\[ L^{-1} \left[ \frac{2s^2 + 5s - 4}{s^3 + s^2 - 2s} \right] = 2e^{-2t} + e^t \]

5. Evaluate: \[ L^{-1} \left[ \frac{4s + 5}{\xi + 1} + \frac{1}{\xi + 2} \right] \]

Let us take

\[ \frac{4s + 5}{\xi + 1} + \frac{1}{\xi + 2} = \frac{A}{\xi + 1} + \frac{B}{\xi + 2} + \frac{C}{\xi + 1} \]

Then \[ 4s + 5 = A(s + 2) + B(s + 1)(s + 2) + C(s + 1)^2 \]

For \( s = -1 \), we get \( A = 1 \), for \( s = -2 \), we get \( C = -3 \)
Comparing the coefficients of $s^2$, we get $B + C = 0$, so that $B = 3$. Using these values in (1),

$$\frac{4s+5}{\zeta+s+2} = \frac{1}{\zeta+s+2} + \frac{3}{s+2}$$

we get

$$\text{Hence } L^{-1}\left[\frac{4s+5}{\zeta+s+2}\right] = e^{-t} L^{-1}\left[\frac{1}{s^2}\right] + 3e^{-t} L^{-1}\left[\frac{1}{s}\right] - 3e^{-2t} L^{-1}\left[\frac{1}{s}\right]$$

$$= te^{-t} + 3e^{-t} - 3e^{-2t}$$

5. Evaluate: $L^{-1}\left[\frac{s^3}{s^4-a^4}\right]$

Let

$$\frac{s^3}{s^4-a^4} = \frac{A}{s-a} + \frac{B}{s+a} + \frac{Cs+D}{s^2+a^2} \quad (I)$$

Hence $s^3 = A(s+a)(s^2+a^2) + B(s-a)(s^2+a^2) + (Cs+D)(s^2-a^2)$

For $s = a$, we get $A = \frac{1}{4}$; for $s = -a$, we get $B = \frac{1}{4}$; comparing the constant terms, we get $D = a(A-B) = 0$; comparing the coefficients of $s^3$, we get $1 = A + B + C$ and so $C = \frac{1}{2}$. Using these values in (I), we get

$$\frac{s^3}{s^4-a^4} = \frac{1}{4} \left[ \frac{1}{s-a} + \frac{1}{s+a} \right] + \frac{1}{2} \frac{s}{s^2+a^2}$$

Taking inverse transforms, we get

$$L^{-1}\left[\frac{s^3}{s^4-a^4}\right] = \frac{1}{4} \left[ at + e^{-at} - \frac{1}{2} \cos at \right]$$

$$= \frac{1}{2} \left[ \cos hat + \cos at \right]$$

6. Evaluate: $L^{-1}\left[\frac{s}{s^4+s^2+1}\right]$

Consider

$$\frac{s}{s^4+s^2+1} = \frac{s}{s^2+s+1} + \frac{s}{s^2-s+1} + \frac{1}{2} \left[ \frac{2s}{s^2+s+1} \right]$$
\[
\frac{1}{2} \left[ \frac{t^2 + s + 1}{t^2 + s + 1} - \frac{t^2 - s + 1}{t^2 - s + 1} \right] \\
\frac{1}{2} \left[ \frac{1}{t^2 - s + 1} - \frac{1}{t^2 + s + 1} \right] \\
\frac{1}{2} \left[ \frac{1}{\frac{1}{2} + \frac{3}{4}} - \frac{1}{\frac{1}{2} + \frac{3}{4}} \right] \\
\]

Therefore

\[
L^{-1} \left[ \frac{s}{s^4 + s^2 + 1} \right] = \frac{1}{2} \left[ \frac{1}{\frac{1}{2}} L^{-1} \left[ \frac{1}{s^2 + \frac{3}{4}} \right] - e^{\frac{1}{2}t} L^{-1} \left[ \frac{1}{s^2 + \frac{3}{4}} \right] \right] \\
= \frac{1}{2} \left[ e^{\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t - e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t \right] \\
= \frac{2}{\sqrt{3}} \sin \left( \frac{\sqrt{3}}{2} t \right) \sin h \left( \frac{t}{2} \right) \\
\]

**Evaluation of** \( L^{-1}[e^{as} F(s)] \)

We have, if \( L[f(t)] = F(s) \), then \( L[f(t-a) H(t-a)] = e^{as} F(s) \), and so

\( L^{-1}[e^{as} F(s)] = f(t-a) H(t-a) \)
Examples

(1) Evaluate: \( L^{-1}\left[ \frac{e^{-as}}{s - 2} \right] \)

Here

\[ a = 5, \; F(s) = \frac{1}{s - 2} \]

Therefore \( f(t) = L^{-1}F(s) = L^{-1}\left( \frac{1}{s - 2} \right) = e^{2r} L^{-1}\left( \frac{1}{s^4} \right) = \frac{e^{2rt^3}}{6} \)

Thus

\[ L^{-1}\left( \frac{e^{-as}}{s - 2} \right) = f(t-a)H(t-a) \]

\[ = \frac{e^{2t-a}e^{-5}}{6} H(t-5) \]

(2) Evaluate: \( L^{-1}\left[ \frac{e^{-\pi s}}{s^2 + 1} + \frac{se^{-2\pi s}}{s^2 + 4} \right] \)

Given: \( f_1(-\pi \right) H(-\pi) + f_2(-2\pi \right) H(-2\pi) \)  

Here \( f_1(t) = L^{-1}\left( \frac{1}{s^2 + 1} \right) = \sin t \)

\( f_2(t) = L^{-1}\left( \frac{s}{s^2 + 4} \right) = \cos 2t \)

Now relation (1) reads as

Given = \( \sin (-\pi \right) H(-\pi) \cos 2(-2\pi \right) H(-2\pi) \)

\[ = -\cos t \; H(-\pi) \cos (t \right) H(-2\pi) \]
Inverse transform of logarithmic functions

We have, if $L f(t) = F(s)$, then $L \left[ \frac{d}{ds} F(s) \right] = t f(t)$

Hence

Examples:

(1) Evaluate: $L^{-1} \log \left( \frac{s + a}{s + b} \right)$

Let $F(s) = \log \left( \frac{s + a}{s + b} \right) = \log \left( 1 + \frac{a}{s + b} \right)$

Then $- \frac{d}{ds} F(s) = \left[ - \frac{1}{s + a} - \frac{1}{s + b} \right]$,

So that $L^{-1} \left[ - \frac{d}{ds} F(s) \right] = - \left[ e^{-at} - e^{-bt} \right]$,

or $t f(t) = e^{-bt} - e^{-at}$

Thus $f(t) = \frac{e^{-bt} - e^{-at}}{b}$

(2) Evaluate $L^{-1} \tan^{-1} \left( \frac{a}{s} \right)$

Let $F(s) = \tan^{-1} \left( \frac{a}{s} \right)$

Then $- \frac{d}{ds} F(s) = \left[ \frac{a}{s^2 + a^2} \right]$
or \( L^{-1}\left[ -\frac{d}{ds} F(\zeta) \right] = \sin at \) so that

or \( t f(\zeta) = \sin at \)

\( f(\zeta) = \frac{\sin at}{a} \)

**Inverse transform of** \( \begin{bmatrix} F \\ s \end{bmatrix} \)

(1) **Evaluate**: \( L^{-1}\left[ \frac{1}{s(\zeta^2 + a^2)} \right] \)

Let us denote \( F(\zeta) = \frac{1}{s(\zeta^2 + a^2)} \) so that

\( f(t) = L^{-1}F(\zeta) = \frac{\sin at}{a} \)

Then \( L^{-1} = \frac{1}{s(\zeta^2 + a^2)} \)

\( L^{-1}F(\zeta) = \int_0^t \frac{\sin at}{a} \, dt = \frac{-\cos at}{a^2} \)

**Convolution Theorem:**

If \( L^{-1}\{F(s)\} = f(t) \) and \( L^{-1}\{G(s)\} = g(t) \)

then \( L^{-1}\{F(s)G(s)\} = \int_0^t f(u)g(t-u) \, du \) ... (1)

**Proof.** Since \( L^{-1}F(s) = f(t) \) and \( L^{-1}\{G(s)\} = g(t) \)

we have \( F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt \)
and 

\[ G(s) = L\{g(t)\} = \int_{0}^{\infty} e^{-st} g(t) \, dt \]

To prove (1), it is sufficient to prove that

\[ L\left\{ \int_{0}^{t} f(u) g(t-u) \, du \right\} = F(s) \cdot G(s) \]  ...(2)

Consider

\[ L\left\{ \int_{0}^{t} f(u) g(t-u) \, du \right\} = \int_{0}^{\infty} e^{-st} \left\{ \int_{0}^{t} f(u) g(t-u) \, du \right\} \, dt \]

\[ = \int_{t=0}^{\infty} \int_{u=0}^{t} e^{-st} f(u) g(t-u) \, du \, dt \]  ...(3)

**Fig. 8.1**

The domain of integration for the above double integral is from \( u = 0 \) to \( u = t \) and \( t = 0 \) to \( t = \infty \) which is as shown in Fig. 8.1.

The double integral given in the R.H.S. of equation (3) indicates that we integrate first parallel to \( u \)-axis and then parallel to \( t \)-axis.

We shall now change the order of integration parallel to \( t \)-axis the limits being \( t = u \) to \( t = \infty \) and parallel to \( u \)-axis the limits being \( u = 0 \) to \( u = \infty \).

.: From equation (3), we get

\[ L\left\{ \int_{0}^{t} f(u) g(t-u) \, du \right\} = \int_{0}^{\infty} f(u) \left\{ \int_{u}^{\infty} e^{-st} g(t-u) \, dt \right\} \, du \]

\[ = \int_{0}^{\infty} f(u) e^{-su} \left\{ \int_{u}^{\infty} e^{-s(t-u)} g(t-u) \, dt \right\} \, du \]
Using Convolution theorem find the inverse Laplace transforms of the following

(i) \( \frac{1}{s^2 (s+1)^2} \)

Solution

Let \( F(s) = \frac{1}{(s+1)^2} \), \( G(s) = \frac{1}{s^2} \)

Then

\[ L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t} = f(t) \text{ (say)} \]

\[ L^{-1}\{G(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} = t = g(t) \text{ say} \]

Then by Convolution theorem, we have

\[ L^{-1}\{F(s) G(s)\} = \int_0^t f(u) g(t-u) \, du \]

\[ L^{-1}\left\{\frac{1}{s^2 (s+1)^2}\right\} = \int_0^t u e^{-u} (t-u) \, du = \int_0^t (ut-u^2) e^{-u} \, du \]
(2) Evaluate: \[ L^{-1} \left[ \frac{1}{s^2 (s+a)} \right] \]

we have \[ L^{-1} \frac{1}{s (s+a)} = e^{-at} t \]

Hence \[ L^{-1} \frac{1}{s^2 (s+a)} = \int_0^t e^{-at} dt \]

\[ = \frac{1}{a^2} \left[ -e^{-at} + \frac{t}{a} \right] \text{ on integration by parts.} \]

Using this, we get

\[ L^{-1} \frac{1}{s^2 (s+a)} = \frac{1}{a^2} \left[ -\frac{t}{a} e^{-at} \right] \]

\[ = \frac{1}{a^2} \left[ t + e^{-at} + \frac{2}{a^2} - 1 \right] \]

Inverse transform of \( F(s) \) by using convolution theorem

We have, if \( L(t) = F(s) \) and \( Lg(t) = G(s) \), then

\[ L \left( f(t) * g(t) \right) = Lf(t) \cdot Lg(t) = F(s) G(s) \text{ and so} \]

\[ L^{-1} \left[ F(s) G(s) \right] = f(t) * g(t) = \int_0^t f(\xi - u) g(\xi) d\xi \]

This expression is called the convolution theorem for inverse Laplace transform.
Examples
Employ convolution theorem to evaluate the following:

(1) $L^{-1}\left[ \frac{1}{s + a} \right]$

Let us denote $F(s) = \frac{1}{s + a}$, $G(s) = \frac{1}{s + b}$

Taking the inverse, we get $f(t) = e^{at}$, $g(t) = e^{-bt}$

Therefore, by convolution theorem,

$$L^{-1}\left[ \frac{1}{s + a} \right] = \int_0^t e^{-at} \left( e^{-bu} - 1 \right) du$$

$$= e^{-at} \left[ \frac{e^{-bt} - 1}{a - b} \right]$$

$$= \frac{e^{-bt} - e^{-at}}{a - b}$$

(2) $L^{-1}\left[ \frac{s}{s^2 + a^2} \right]$

Let us denote $F(s) = \frac{1}{s^2 + a^2}$, $G(s) = \frac{s}{s^2 + a^2}$

Then

$f(t) = \frac{\sin at}{a}$, $g(t) = \cos at$

Hence by convolution theorem,

$$L^{-1}\left[ \frac{s}{s^2 + a^2} \right] = \int_0^t \frac{1}{a} \sin at \cos au du$$

$$= \frac{1}{a} \int_0^t \frac{\sin at + \sin (t - 2au)}{2} du$$

by using compound angle formula

$$= \frac{1}{2a} \left[ \frac{u \sin at - \cos (t - 2au)}{-2a} \right]_0^t = \frac{t \sin at}{2a}$$
(3) \( L^{-1} \frac{s}{s^2 + 1} \)

Here

\[
F(s) = \frac{1}{s-1}, \quad G(s) = \frac{s}{s^2 + 1}
\]

Therefore

\[
f(t) = e^t, \quad g(t) = \sin t
\]

By convolution theorem, we have

\[
L^{-1} \left( \frac{1}{s^2 + 1} \right) = \int_0^t e^{t-u} \sin u \, du = e^t \left[ \frac{e^{-u}}{2} \right]_0^t \sin u - \cos u
\]

\[
= \frac{e^t}{2} \left( \sin t - \cos t \right) - \frac{1}{2} \left( \sin t - \cos t \right)
\]

**LAPLACE TRANSFORM METHOD FOR DIFFERENTIAL EQUATIONS**

As noted earlier, Laplace transform technique is employed to solve initial-value problems. The solution of such a problem is obtained by using the Laplace Transform of the derivatives of function and then the inverse Laplace Transform.

The following are the expressions for the derivatives derived earlier.

\[
L[f'(t)] = s \, L[f(t)] - f(0)
\]

\[
L[f''(t)] = s^2 \, L[f(t)] - s \, f(0) - f'(0)
\]

\[
L[f'''(t)] = s^3 \, L[f(t)] - s^2 \, f(0) - s \, f'(0) - f''(0)
\]
1. **Solve by using Laplace transform method**

   \[ y'' + y = t e^{-t}, \quad y(0) = 2 \]

   Taking the Laplace transform of the given equation, we get

   \[
   \begin{align*}
   \mathcal{L}\{y''\} + \mathcal{L}\{y\} &= \frac{1}{s} \quad \Rightarrow \\
   s^2 Y(s) - sy(0) - y'(0) + sY(s) - y(0) &= \frac{1}{s} \\
   \mathcal{L}\{y''\} + \mathcal{L}\{y\} &= \frac{1}{s}.
   \end{align*}
   \]

   so that

   \[ \mathcal{L}\{y\} = \frac{2s^2 + 4s + 3}{s + 1} \]

   Taking the inverse Laplace transform, we get

   \[
   Y(s) = \mathcal{L}^{-1}\left\{\frac{2s^2 + 4s + 3}{s + 1}\right\} = L^{-1}\left[\frac{2(s+1-1) + 4(s+1-1) + 3}{s+1}\right]
   \]

   \[
   = \frac{1}{2} e^{-t} (s + 4)
   \]

   This is the solution of the given equation.

2. **Solve by using Laplace transform method:**

   \[ y'' + 2y' - 3y = \sin 2t, \quad y(0) = y'(0) = 0 \]

   Taking the Laplace transform of the given equation, we get

   \[
   \begin{align*}
   \mathcal{L}\{y''\} - s^2 y(s) - s y'(0) - y(0) - 2 \mathcal{L}\{y'\} + 2 s y(0) - 2 y'(0) &= \frac{1}{s^2 + 4} \\
   \mathcal{L}\{y''\} + 2 \mathcal{L}\{y'\} - 3 \mathcal{L}\{y\} &= \frac{1}{s^2 + 4}.
   \end{align*}
   \]

   Using the given conditions, we get
\[ L \ y(t) \begin{bmatrix} 1 + 2s - 3 \end{bmatrix} = \frac{1}{s^2 + 1} \]

or

\[ L \ y(t) = \frac{1}{s - 1} + 3 \frac{s^2 + 1}{s^2 + 1} \]

or

\[ y(t) = L^{-1} \left[ \frac{1}{s - 1} + 3 \frac{s^2 + 1}{s^2 + 1} \right] \]

\[ = L^{-1} \left[ \frac{A}{s - 1} + \frac{B}{s + 3} + \frac{Cs + D}{s^2 + 1} \right] \]

by using the method of partial sums,

\[ = \frac{1}{8} e^t - \frac{1}{40} e^{-3t} - \frac{1}{10} \cos t + 2 \sin t \]

This is the required solution of the given equation.

3) Employ Laplace Transform method to solve the integral equation.

\[ f(t) = 1 + \int_{0}^{t} f \ \sin \ u \ \ du \]

Taking Laplace transform of the given equation, we get

\[ L f(t) = \frac{1}{s} + L \int_{0}^{t} f \ \sin \ u \ \ du \]

By using convolution theorem, here, we get
\[ L f(t) = \frac{1}{s} + Lf(t) \cdot L \sin t = \frac{1}{s} + \frac{L f(t)}{s^2 + 1} \]

Thus

\[ L f(t) = \frac{s^2 + 1}{s^3} \quad \text{or} \quad f(t) = L^{-1} \left( \frac{s^2 + 1}{s^3} \right) = 1 + \frac{t^2}{2} \]

This is the solution of the given integral equation.

(4) A particle is moving along a path satisfying, the equation \( \frac{d^2 x}{dt^2} + 6 \frac{dx}{dt} + 25x = 0 \) where \( x \) denotes the displacement of the particle at time \( t \). If the initial position of the particle is at \( x = 20 \) and the initial speed is 10, find the displacement of the particle at any time \( t \) using Laplace transforms.

Given equation may be rewritten as

\[ x''(t) + 6x'(t) + 25x(t) = 0 \]

Here the initial conditions are \( x(o) = 20, \ x'(o) = 10. \)

Taking the Laplace transform of the equation, we get

\[ Lx(t) \left[ \frac{d^2 x}{dt^2} + 6 \frac{dx}{dt} + 25x \right] = 0 \quad \text{or} \]

\[ Lx(t) = \frac{20s + 130}{s^2 + 6s + 25} \]

so that

\[ x(t) = L^{-1} \left[ \frac{20s + 130}{\xi + 3 \xi^2 + 16} \right] \]

\[ = L^{-1} \left[ \frac{20 \xi + 70}{\xi + 3 \xi^2 + 16} \right] \]

\[ = 20 L^{-1} \left[ \frac{s + 3}{\xi + 3 \xi^2 + 16} \right] + 70 L^{-1} \left[ \frac{1}{\xi + 3 \xi^2 + 16} \right] \]
= 20 e^{-3t} \cos 4t + 35 \frac{e^{-3t}}{2} \sin 4t

This is the desired solution of the given problem.

(5) A voltage $Ee^{-at}$ is applied at $t = 0$ to a circuit of inductance $L$ and resistance $R$. Show that the current at any time $t$ is

$$\frac{E}{R - aL} \left[ e^{-at} - e^{-\frac{Rt}{L}} \right]$$

The circuit is an LR circuit. The differential equation with respect to the circuit is

$$L \frac{di}{dt} + Ri = E(t)$$

Here $L$ denotes the inductance, $i$ denotes current at any time $t$ and $E(t)$ denotes the E.M.F.

It is given that $E(t) = E e^{-at}$. With this, we have

Thus, we have

$$L \frac{di}{dt} + Ri = E e^{-at} \quad \text{or}$$

$$Li'(t) + R i(t) = E e^{-at}$$

$$L \begin{bmatrix} T \\ T \end{bmatrix} i(t) + R \begin{bmatrix} T \\ T \end{bmatrix} i(t) = E \begin{bmatrix} T \\ T \end{bmatrix} \left( e^{-at} \right) \quad \text{or}$$

Taking Laplace transform ($L_T$) on both sides, we get

$$L \begin{bmatrix} T \\ T \end{bmatrix} i(t) - i(0) + R \begin{bmatrix} T \\ T \end{bmatrix} i(t) = E \frac{1}{s + a}$$

Since $i(0) = 0$, we get

$$L_T i(t) L + R \frac{E}{s + a} \quad \text{or}$$
Taking inverse transform L, we get

\[ i(t) = L_t^{-1} \frac{E}{(s + a)(sL + R)} \]

\[ = \frac{E}{R - aL} \left[ \frac{1}{s + a} L^{-1} - L^{-1} \frac{1}{sL + R} \right] \]

Thus

\[ i(t) = \frac{E}{R - aL} \left[ e^{-at} - e^{-\frac{Rt}{L}} \right] \]

This is the result as desired.

(6) Solve the simultaneous equations for x and y in terms of t given \( \frac{dx}{dt} + 4y = 0 \),

\( \frac{dy}{dt} - 9x = 0 \) with \( x(0) = 2, y(0) = 1 \).

Taking Laplace transforms of the given equations, we get

\[ \begin{cases} Lx(t) - x(0) + 4Ly(t) = 0 \\ -9Lx(t) + Ly(t) - y(0) = 0 \end{cases} \]

Using the given initial conditions, we get

\[ \begin{align*} sLx(t) + 4Ly(t) &= 2 \\ -9Lx(t) + 5Ly(t) &= 1 \end{align*} \]

Solving these equations for \( Ly(t) \), we get

\[ Ly(t) = \frac{s + 18}{s^2 + 36} \]

so that

\[ y(t) = L_t^{-1} \left[ \frac{s}{s^2 + 36} + \frac{18}{s^2 + 36} \right] \]

\[ = \cos 6t + 3 \sin 6t \quad (1) \]
Using this in $\frac{dy}{dt} - 9x = 0$, we get

$$x(t) = \frac{1}{9} \left[ 6 \sin 6t + 18 \cos 6t \right]$$

or

$$x(t) = \frac{2}{3} \left[ \cos 6t - \sin 6t \right]$$

(2)

(1) and (2) together represents the solution of the given equations.